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On a solvable system of difference equations of kth order

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ABSTRACT

It is shown that the next system of k difference equations

$$x_n^{(j)} = \frac{a_n^{(j)} x_{n-k}^{(j)}}{b_n^{(j)} \prod_{i=1}^{k} x_{n-i}^{(\sigma(j+i-1))} + c_n^{(j)}}, \quad n \in \mathbb{N}_0, \ j = \overline{1,k},$$

where $a_n^{(j)}, b_n^{(j)}, c_n^{(j)}, n \in \mathbb{N}_0, j = \overline{1,k}$, and initial values $x_{-i}^{(j)}, i,j \in \{1,\ldots,k\}$, are real numbers, and where $\sigma : \mathbb{N} \to \{1,\ldots,k\}$ is defined by $\sigma(km+j) = j+1, j = \overline{1,k-1}, \sigma(km+k) = 1$, $m \in \mathbb{N}_0$, can be solved in closed form in an elegant way. Some applications of obtained formulas, for the case when the sequences $a_n^{(j)}, b_n^{(j)}$ and $c_n^{(j)}$ are k-periodic are also given.

1. Introduction

Studying systems of difference equations has attracted some attention recently (see, for example, [6,8–12,15,21,24–26,28–32]). There has been also some renewed interest in equations and systems which can be solved in closed form, as well as in their applications (see, for example, [1–5,14,16,17,21,22,24,26–33]).

Here we study the following system of k difference equations

$$x_n^{(j)} = \frac{a_n^{(j)} x_{n-k}^{(j)}}{b_n^{(j)} \prod_{i=1}^{k} x_{n-i}^{(\sigma(j+i-1))} + c_n^{(j)}}, \quad n \in \mathbb{N}_0, \ j = \overline{1,k}, \tag{1}$$

where sequences $(a_n^{(j)})_{n\in\mathbb{N}_0}, (b_n^{(j)})_{n\in\mathbb{N}_0}, (c_n^{(j)})_{n\in\mathbb{N}_0}, j = \overline{1,k}$, and initial values $x_{-i}^{(j)}, i,j \in \{1,\ldots,k\}$, are real, and where the mapping $\sigma : \mathbb{N} \to \{1,\ldots,k\}$ is defined by

$$\sigma(km+j) = j+1, \quad j = \overline{1,k-1}, \quad \sigma(km+k) = 1, \quad m \in \mathbb{N}_0.$$

In [21] we solved the system (1) with constant coefficients for the case $k = 2$, in closed form, while in [24] we did the same for the case $k = 3$. In [31] we solved the system (1) with variable coefficients for the case $k = 3$ in closed form, and presented numerous applications of obtained formulas. The main idea in [21,24,31] stems from our paper [16] where a related scalar equation is solved in closed form by a suitable transformation (for some extensions of the results in [16], see [22]). Papers [21,24,31] motivated us to consider a system which contains into itself all the systems appearing therein. System (1) is a natural extension of these ones. For related equations, systems and results see also [1,2,7,13,14,17–20,23,27,29,30,32,33]. Our aim here is to solve system (1) in closed form, that is, we find formulas for all well-defined solutions of the system (i.e. those ones for which $b_n^{(j)} \prod_{i=1}^{k} x_{n-i}^{(\sigma(j+i-1))} + c_n^{(j)} \neq 0$ for every $n \in \mathbb{N}_0$ and $j = \overline{1,k}$), in closed form.

If $k,l \in \mathbb{Z}, k < l$, then by $\overline{k,l}$ we denote the set $\{k, k+1, \ldots, l\} \subset \mathbb{Z}$.

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2. Case $a_n^{(j)} = 0$ for some $j \in \{1, \ldots, k\}$ and all $n \in N_0$

If $a_n^{(j_0)} = 0$, for some $j_0 \in \{1, \ldots, k\}$ and all $n \in \mathbb{N}_0$, then

$$x_n^{(j_0)} = 0, \quad n \in \mathbb{N}_0. \tag{2}$$

Using (2) in (1) we have that

$$x_n^{(j)} = \frac{a_n^{(j)}}{c_n^{(j)}} x_{n-k}^{(j)},$$

for $n \geqslant k + j_0 - j$, when $j = \overline{j_0 + 1, k}$, and for $n \geqslant j_0 - j$, when $j = \overline{1, j_0 - 1}$, and when $c_n^{(j)} \neq 0$, for $n \in \mathbb{N}_0$ and $j \in \{1, \ldots, k\} \setminus \{j_0\}$.

Hence

$$x_{km+l}^{(j)} = x_l^{(j)} \prod_{i=1}^{m} \frac{a_{ki+l}^{(j)}}{c_{ki+l}^{(j)}}, \quad m \in \mathbb{N}_0, \ j = \overline{1, k}, \tag{3}$$

for $l \geqslant j_0 - j$, when $j = \overline{j_0 + 1, k}$, and for $l \geqslant j_0 - j - k$, when $j = \overline{1, j_0 - 1}$, and when $c_n^{(j)} \neq 0$, for every $n \in \mathbb{N}_0$ and $j \in \{1, \ldots, k\} \setminus \{j_0\}$.

3. Case $a_n^{(j)} \neq 0$ for all $n \in N_0$ and $j = \overline{1, k}$

Here we consider system (1) in the case when $a_n^{(j)} \neq 0$ for every $n \in \mathbb{N}_0$ and $j = \overline{1, k}$.

First note that if $x_{n_0}^{(j_0)} = 0$ for some $n_0 \in \mathbb{N}_0$ and $j_0 \in \{1, \ldots, k\}$, then by using (1) we obtain $x_{n_0 - kl}^{(j_0)} = 0$, for every $l \in \mathbb{N}_0$ such that $n_0 - kl \geqslant -k$. Thus $x_{-i_0}^{(j_0)} = 0$ for some $i_0 \in \{1, \ldots, k\}$. On the other hand, if $x_{-i_0}^{(j_0)} = 0$ for some $i_0, j_0 \in \{1, \ldots, k\}$, then by using (1) it follows that $x_{km-i_0}^{(j_0)} = 0$, for every $m \in \mathbb{N}_0$.

Now, note that since $a_n^{(j)} \neq 0$ for $n \in \mathbb{N}_0$ and $j = \overline{1, k}$, system (1) can be written in the form

$$x_n^{(j)} = \frac{x_{n-k}^{(j)}}{\hat{b}_n^{(j)} \prod_{i=1}^{k} x_{n-i}^{(\sigma(j+i-1))} + \hat{c}_n^{(j)}}, \quad n \in \mathbb{N}_0, \ j = \overline{1, k},$$

where $\hat{b}_n^{(j)} = b_n^{(j)} / a_n^{(j)}, \hat{c}_n^{(j)} = c_n^{(j)} / a_n^{(j)}, n \in \mathbb{N}_0, j = \overline{1, k}$.

So, for the notational simplicity we may study the next system

$$x_n^{(j)} = \frac{x_{n-k}^{(j)}}{b_n^{(j)} \prod_{i=1}^{k} x_{n-i}^{(\sigma(j+i-1))} + c_n^{(j)}}, \quad n \in \mathbb{N}_0, \ j = \overline{1, k}, \tag{4}$$

with the notation as in (1) except for $a_n^{(j)}$, which we assume that $a_n^{(j)} = 1$ for every $n \in \mathbb{N}_0$ and $j = \overline{1, k}$.

3.1. Main case

Here we assume $a_n^{(j)} \neq 0$ for $n \in \mathbb{N}_0, j = \overline{1, k}$, and $x_{-i}^{(j)} \neq 0, i, j \in \{1, \ldots, k\}$. Recall that we may assume that $a_n^{(j)} = 1$, for $n \in \mathbb{N}_0$ and $j = \overline{1, k}$.

Multiplying the jth equation in system (4) by $\prod_{i=1}^{k-1} x_{n-i}^{(\sigma(j+i-1))}$, and then using in such obtained system the change of variables

$$u_n^{(j)} = \frac{1}{\prod_{i=0}^{k-1} x_{n-i}^{(\sigma(j+i-1))}}, \quad n \geqslant -1, \ j = \overline{1, k}, \tag{5}$$

the system becomes

$$u_n^{(j)} = c_n^{(j)} u_{n-1}^{(j+1)} + b_n^{(j)}, \quad n \in \mathbb{N}_0, \ j = \overline{1, k}. \tag{6}$$

From (6) we have that for $n \geqslant k - 1$

$$u_n^{(j)} = \left(\prod_{i=0}^{k-1} c_{n-i}^{(\sigma(j+i-1))} \right) u_{n-k}^{(j)} + \sum_{i=0}^{k-1} b_{n-i}^{(\sigma(j+i-1))} \prod_{s=0}^{i-1} c_{n-s}^{(\sigma(j+s-1))}, \tag{7}$$

($u_{-1}^{(j)}, j = \overline{1, k}$, are obtained from (5)).

From (7) we see that $(u_{km+i}^{(j)})_{m \in \mathbb{N}_0}, j = \overline{1, k}$, $i \in \{-1, 0, 1, \ldots, k - 2\}$, are solutions of the following difference equations

$$u_{km+i}^{(j)} = \left(\prod_{s=0}^{k-1} c_{km+i-s}^{(\sigma(j+s-1))} \right) u_{k(m-1)+i}^{(j)} + \sum_{l=0}^{k-1} b_{km+i-l}^{(\sigma(j+l-1))} \prod_{s=0}^{l-1} c_{km+i-s}^{(\sigma(j+s-1))}, \quad m \in \mathbb{N}. \tag{8}$$