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Initial value problems for neutral fractional differential equations involving a Riemann–Liouville derivative

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ABSTRACT

We present the application of a monotone iterative method to neutral fractional problems. Given are sufficient conditions which guarantee that a neutral fractional differential equations with initial condition has a unique solution. Two examples illustrate the results.

1. Introduction

This paper discusses the existence of solutions of problems:

$$\begin{cases} D^q x(t) & = f(t, D^q x(t), D^q x(\alpha(t)), x(t)), \quad t \in J_0 = (0, T], \quad T > 0, \\ \tilde{x}(0) & = 0, \end{cases} \tag{1}$$

where $D^q x$ denotes a Riemann–Liouville fractional derivative of x with $q \in (0,1), \tilde{x}(0) = t^{1-q}x(t)|_{t=0}$, and

$$H_1: f \in C(J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \quad \alpha \in C(J, J), \quad \alpha(t) \leqslant t \quad \text{on} \quad J = [0, T].$$

If f does not depend on the second and third arguments, then problem (1) is not of neutral type which was considered, for example, in papers [1–11] using the monotone iterative method. In this paper, we also apply this technique to obtain existing results. To do it, we first translate problem (1) to a corresponding functional one, by the substitution $D^q x(t) = y(t)$. Note that a similar technique has been discussed for fractional problems considered in paper [10]. To the author's knowledge, it is a first paper when the monotone iterative method has been applied to neutral fractional differential problems. If $q = 1$, then problem (1) reduces to a corresponding neutral problem in ordinary differential equations.

2. Existence results for problem (1)

Put $D^q x(t) = u(t)$. Then problem (1) takes the form

$$u(t) = f(t, u(t), u(\alpha(t)), Bu(t)) \equiv Fu(t), \quad t \in J = [0, T], \tag{2}$$

where operator B is defined by

$$Bu(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s) ds. \tag{3}$$

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Let us introduce the following definition. We say that $u \in C(J, \mathbb{R})$ is called a lower solution of (2) if

$$u(t) \leqslant Fu(t), \quad t \in J,$$

and it is an upper solution of (2) if the above inequality is reversed.

Now we formulate conditions under which problem (2) has extremal solutions in a corresponding sector bounded by lower and upper solutions of problem (2).

Theorem 1. *Let assumption H_1 hold. Let $y_0, z_0 \in C(J, \mathbb{R})$ be the lower and upper solutions of problem (2), respectively and $y_0(t) \leqslant z_0(t),\ t \in J$. In addition, let us assume that the following assumptions hold:*

H_2: f is nondecreasing with respect to the last two variables,
H_3: there exists a constant $K > -1$, such that

$$f(t, u, v_1, v_2) - f(t, \bar{u}, v_1, v_2) \leqslant K(\bar{u} - u)$$

for $y_0 \leqslant u \leqslant \bar{u} \leqslant z_0$.

Then problem (2) has, in the sector $[y_0, z_0]_*$, minimum and maximum solutions, where

$$[y_0, z_0]_* = \{v \in C(J, \mathbb{R}) : y_0(t) \leqslant v(t) \leqslant z_0(t),\ t \in J\}.$$

Proof. Let

$$\begin{cases} y_{n+1}(t) &= Fy_n(t) - K[y_{n+1}(t) - y_n(t)],\ t \in J, \\ z_{n+1}(t) &= Fz_n(t) - K[z_{n+1}(t) - z_n(t)],\ t \in J \end{cases}$$

for $n = 0, 1, \ldots$, with the operator F defined as in formula (2).

Observe that functions y_1, z_1 are well defined. First, we prove that

$$y_0(t) \leqslant y_1(t) \leqslant z_1(t) \leqslant z_0(t), \quad t \in J. \tag{4}$$

Put $p = y_0 - y_1,\ q = z_1 - z_0$. It leads to

$$p(t) \leqslant Fy_0(t) - Fy_0(t) - Kp(t) = -Kp(t)$$
$$q(t) \leqslant Fz_0(t) - Fz_0(t) - Kq(t) = -Kq(t).$$

This shows that $y_0(t) \leqslant y_1(t),\ z_1(t) \leqslant z_0(t),\ t \in J$. Now, we put $p = y_1 - z_1$. In view of assumptions H_2, H_3 we have

$$p(t) = Fy_0(t) - Fz_0(t) - K[y_1(t) - y_0(t) - z_1(t) + z_0(t)] \leqslant -Kp(t).$$

Hence, $y_1(t) \leqslant z_1(t)$ on J. It proves relation (4).

In the next step we show that y_1, z_1 are the lower and upper solutions of problem (2). Note that

$$y_1(t) = Fy_0(t) - Fy_1(t) + Fy_1(t) - K[y_1(t) - y_0(t)] \leqslant Fy_1(t),$$
$$z_1(t) = Fz_0(t) - Fz_1(t) + Fz_1(t) - K[z_1(t) - z_0(t)] \geqslant Fz_1(t),$$

by assumptions H_2, H_3. This proves that y_1, z_1 are the lower and upper solutions of problem (2).

Using the mathematical induction, we can show that

$$y_0(t) \leqslant y_1(t) \leqslant \ldots \leqslant y_n(t) \leqslant y_{n+1}(t) \leqslant z_{n+1}(t) \leqslant z_n(t) \leqslant \cdots \leqslant z_1(t) \leqslant z_0(t)$$

for $t \in J$ and $n = 0, 1, \ldots$.

It is easy to see that sequences $\{y_n, z_n\}$ converge uniformly and monotonically to the limit functions y and z, respectively; where y and z are solutions of the following problems:

$$y(t) = Fy(t), \quad t \in J,$$
$$z(t) = Fz(t), \quad t \in J$$

with $y_0 \leqslant y \leqslant z \leqslant z_0$.

Now, we need to show that y and z are extremal solutions of problem (2) in the sector $[y_0, z_0]_*$. Let v be any solution of problem (2) such that $y_0 \leqslant v \leqslant z_0$. Put $p = y_1 - v,\ q = v - z_1$. Then, in view of assumptions H_2 and H_3, we see that

$$p(t) = Fy_0(t) - K[y_1(t) - y_0(t)] - Fv(t) \leqslant -Kp(t),$$
$$q(t) = Fv(t) - Fz_0(t) + K[z_1(t) - z_0(t)] \leqslant -Kq(t).$$

This shows that $y_1 \leqslant v \leqslant z_1$. By induction, we can show that

$$y_n \leqslant v \leqslant z_n.$$

Now, if $n \to \infty$, then we have the assertion of this theorem.

Our next theorem concerns the case when problem (2) has a unique solution.