Applied Mathematics and Computation 219 (2013) 7847–7852

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On subordination-preserving theorem

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ARTICLE INFO

Keywords:
Hadamard product
Dual set
Convex functions
Prestarlike functions
Subordination
Preserving operator

ABSTRACT

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc on the complex plane $\mathbb{C}$. If the operator $I:\mathcal{H}\to\mathcal{H}$ satisfies

$$f(z)\prec g(z) \;\Rightarrow\; I[f](z)\prec I[g](z)$$

for all $f,g\in\mathcal{H}$, then it is called subordination-preserving operator on the class $\mathcal{H}$. In this paper we present a new result of this form which generalize some of the earlier and is connected with the Wilf's conjecture. The applications of main result are also presented.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathcal{U}=\{z:\ |z|<1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the class of all functions $f\in\mathcal{H}$ normalized by $f(0)=0, f'(0)=1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ whose members are univalent in $\mathcal{U}$. For $\gamma\geqslant 0, \alpha\geqslant 0$ let us denote

$$\mathcal{K}(0,\gamma)=\left\{g\in\mathcal{H}:g(0)=1,\ \Re e\frac{zg'(z)}{g(z)}>-\frac{\gamma}{2},\ z\in\mathcal{U}\right\}, \tag{1.1}$$

$$\mathcal{H}^{\alpha}=\left\{h^{\alpha}\in\mathcal{H}:h^{\alpha}(0)=1,\ \Re e(e^{i\gamma}h(z))>0,\ z\in\mathcal{U},\ \text{for certain } \gamma\in\mathbb{R}\right\}. \tag{1.2}$$

Now let

$$\mathcal{K}(\alpha,\beta)=\mathcal{H}^{\alpha}\cdot\mathcal{K}(0,\beta-\alpha),\quad 0\leqslant\alpha\leqslant\beta, \tag{1.3}$$

and

$$\mathcal{K}(\alpha,\beta)=\{1/f:f\in\mathcal{K}(\beta,\alpha)\},\quad 0\leqslant\beta\leqslant\alpha, \tag{1.4}$$

where $X\cdot Y$ is the direct product of $X,Y\subset\mathcal{H}$

$$X\cdot Y=\{f:f=gh,\ g\in X,\ h\in Y\}.$$

The class $\mathcal{K}(\alpha,\beta)$ is called the Kaplan class of type α,β, because $\mathcal{K}(1,3)$ is the class of derivatives of the so-called close-to-convex functions $\mathcal{C}_0$, first introduced by Kaplan in [3]. The functions in $\mathcal{C}_0$ form an important subclass of $\mathcal{S}$. The notion of Kaplan classes unifies also other various geometrically defined classes of functions. It is clear from (1.1) that the class $\mathcal{S}^*_\alpha$ of starlike functions of order $\alpha<1$ may be defined as

$$f\in\mathcal{S}^*_\alpha \Longleftrightarrow \frac{f}{z}\in\mathcal{K}(0,2-2\alpha),\quad \mathcal{S}^*_\alpha=\left\{f\in\mathcal{A}:\ \Re e\frac{zf'(z)}{f(z)}>\alpha,\ z\in\mathcal{U}\right\}.$$

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.01.068

The class $\mathcal{S}^*_\alpha$ and the class $\mathcal{K}_\alpha$ of convex functions of order $\alpha < 1$

$$\mathcal{K}_\alpha := \left\{ f \in \mathcal{A} : \ \Re e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in \mathcal{U} \right\} = \{ f \in \mathcal{A} : \ zf' \in \mathcal{S}^*_\alpha \} \tag{1.5}$$

introduced Robertson in [8]. If $\alpha \in [0;1)$, then a function in either of these sets is univalent, if $\alpha < 0$ it may fail to be univalent. In particular we denote $\mathcal{S}^*_0 = \mathcal{S}^*, \mathcal{K}_0 = \mathcal{K}$, the classes of starlike and convex functions, respectively.

For $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ and $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ the Hadamard product (or convolution) is defined by $(f * g)(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \cdots$. The convolution has the algebraic properties of ordinary multiplication. Many of convolution properties were studied by Rusheweyh in [10] and have found a lot of applications in various fields. For $V \subset \mathcal{A}$ denote the dual set V^* as

$$V^* = \left\{ g \in \mathcal{A} : \forall f \in V, \frac{f(z)}{z} * \frac{g(z)}{z} \neq 0 \ \text{ in } \mathcal{U} \right\},$$

and $V^{**} = (V^*)^*$, the second dual. Let $U \subset \mathcal{H}$. Then T is called a test set for U (written $T \sim> U$) if

$$T \subset U \subset T^{**}. \tag{1.6}$$

It is known [10, p. 33] that for $\alpha \geqslant 1$, $\beta \geqslant 1$

$$\mathcal{T}(\alpha,\beta) \sim> \mathcal{K}(\alpha,\beta), \tag{1.7}$$

where

$$\mathcal{T}(\alpha,\beta) = \left\{ \frac{(1+xz)^{[\alpha]}(1+yz)^{\alpha-[\alpha]}}{(1+uz)^{\beta}} : x, y, z \in \overline{\mathcal{U}} \right\}. \tag{1.8}$$

The functions f such that

$$\frac{f}{z} \in \mathcal{T}(1, 3-2\alpha)^* \tag{1.9}$$

form an important class called the class of prestarlike functions of order $\alpha \leqslant 1$ denoted by $\mathcal{R}_\alpha$. By (1.7) the class $\mathcal{R}_\alpha$ is also strongly related to the class $\mathcal{K}(\alpha,\beta)$. Some calculations show that $f \in \mathcal{R}_\alpha$ whenever $f \in \mathcal{A}$ and f satisfies

$$f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{S}^*_\alpha \ \text{ when } \ \alpha < 1, \tag{1.10}$$

while

$$\Re e \frac{f(z)}{z} > \frac{1}{2} \ \text{ for } \ \alpha = 1. \tag{1.11}$$

The prestarlike functions play a central role in some situations. The special cases $\alpha = 0, 1/2$ give $\mathcal{R}_0 = \mathcal{K}$, $\mathcal{R}_{1/2} = \mathcal{S}^*_{1/2}$. Moreover,

$$\mathcal{K} \subset \mathcal{R}_\alpha \subset \mathcal{R}_\beta \subset \mathcal{R}_1 = \overline{co}\mathcal{K} = \left\{ f \in \mathcal{A} : \Re e \frac{f(z)}{z} > \frac{1}{2} \right\} \ \text{ for } \ \alpha \leqslant \beta \leqslant 1,$$

where $\overline{co}\mathcal{K}$ denotes the closed convex hull of the class of convex functions.

We say that the $f \in \mathcal{H}$ is subordinate to $g \in \mathcal{H}$ in the unit disc $\mathcal{U}$, written $f \prec g$ if and only if there exits an analytic function $w \in \mathcal{H}$ such that $w(0) = 0, |w(z)| < 1$ and $f(z) = g[w(z)]$ for $z \in \mathcal{U}$. Therefore $f \prec g$ in $\mathcal{U}$ implies $f(\mathcal{U}) \subset g(\mathcal{U})$. In particular if g is univalent in $\mathcal{U}$, then

$$f \prec g \quad \Longleftrightarrow \quad [f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U})]. \tag{1.12}$$

2. Preliminary results

We shall use the following lemmas.

Lemma 2.1 ([10, p. 37]). *For $\alpha, \beta \geqslant 1$ let $f \in \mathcal{T}(\alpha,\beta)^*$ and $g \in \mathcal{K}(\alpha - 1, \beta - 1)$. Then, for any $F \in \mathcal{H}$,*

$$\frac{f * gF}{f * g}(\mathcal{U}) \subset \overline{co}(F(\mathcal{U})), \tag{2.1}$$

where $\overline{co}(F(\mathcal{U}))$ denotes the closed convex hull of the set $F(\mathcal{U})$.

The assumptions of the above lemma say that the functions f, g are normalized by $f(0) = g(0) = 1$ but it easy to see that (2.1) remains true if $f(z)/f(0) \in \mathcal{T}(\alpha,\beta)^*$ and $g(z)/g(0) \in \mathcal{K}(\alpha - 1, \beta - 1)$.