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Algebraically computable piecewise linear nodal oscillators

Enrique Ponce *, Javier Ros, Elísabet Vela

Departamento de Matemática Aplicada II, Universidad de Sevilla, Escuela Técnica Superior de Ingeniería, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain

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ABSTRACT

A family of piecewise linear oscillators whose oscillation can be completely characterized by algebraic methods is studied. It represents up to the best of authors's knowledge, the first planar example where all the oscillation properties can be determined for all the values of the bifurcation parameter. In fact, algebraic expressions for coordinates of representative points, period and characteristic multiplier of the corresponding periodic orbit are provided. Thus, the studied family of oscillators deserves to be considered a good benchmark for testing approximate methods of analysis in nonlinear oscillation theory.

The piecewise linear oscillators studied are called nodal oscillators, since their relevant linear parts are of node type, and they are not perturbations of the harmonic oscillator. They represent real models in practice, as it is shown for an electronic circuit modeling a piecewise linear version of the classical Van der Pol oscillator.

1. Introduction and statement of main results

In the last years, there has been an upsurge in the interest of researchers for the study of piecewise linear differential systems, to be referred in what follows as to PWL systems for short. They constitute a class of differential systems which is widely used in modeling many real problems, see for instance [1] and references therein. In this direction of interest, we study here the possible existence of symmetric piecewise linear oscillators whose involved linear parts are of node type. Typically, in explaining the origin of oscillations, it is required to have eigenvalues near the imaginary axis of the complex plane. This is not needed however when dealing with piecewise linear systems. In fact, we will introduce a family of PWL oscillators with all dynamics of node type, which can be called nodal oscillators. In particular, we consider some of them having the outstanding characteristic of being algebraically determinable, that is, all the magnitudes related with the oscillation can be algebraically computed.

We start by considering general planar piecewise linear systems with symmetry respect to the origin and three linearity regions separated by parallel straight lines, under the generic condition of observability. The observability hypothesis is a necessary condition to get a proper planar dynamics and for the appearance of oscillations, see [2], and allows us to build its Liénard-like canonical form. This kind of models are frequent in applications from electronic engineering and nonlinear control systems, where piecewise linear models cannot be considered as idealized ones; they are used in mathematical biology as well, see [3–6], where they constitute approximate models. Since non-smooth piecewise linear characteristics can be considered as the uniform limit of smooth nonlinearities, the global dynamics of smooth models has been sometimes approximated by piecewise linear models and viceversa, as done in [7,8], obtaining a good qualitative agreement between the two modeling approaches. This kind of global linearization by linear pieces emphasizes the importance of PWL systems.

* Corresponding author.
E-mail addresses: eponcem@us.es (E. Ponce), javieros@us.es (J. Ros), elivela@us.es (E. Vela).

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From the three linearity regions, the central one will contain the origin and the other two zones will be referred as to external zones. The key hypothesis to get an algebraically computable piecewise linear oscillation is to impose that the involved spectra have the same proportion between eigenvalues, as done in [9,10]. Thus, as it will be seen, it is possible to convert the transcendental equations that characterize oscillations into algebraic equations. Here, for sake of simplicity, we will choose the ratio 1 : 2. Of course, from Bendixson–Dulac's Theorem we also need for periodic oscillations that the divergence of the vector field, which is constant in each linearity region, have no global constant sign; otherwise self-sustained oscillations are not possible. Thus, we assume that for the external zones we have a dissipative spectrum of the form $\{-\mu, -2\mu\}$, with $\mu > 0$, whilst in the central zone we have a region with the spectrum $\{\eta, 2\eta\}$, obeying the same proportion 1 : 2. This idea has been also exploited in higher dimensions, by using the proportion 1 : 2 : 3, see [9,10].

Our first result says that one needs only to study a one-parameter family to cope with all possible systems with the above characteristics.

Proposition 1. *Consider the family of piecewise linear differential systems*

$$\dot{\mathbf{x}} = A\mathbf{x} + \varphi(c^T\mathbf{x})b, \tag{1}$$

where $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$, A is a 2×2 matrix, $b, c \in \mathbb{R}^2$ and the nonlinearity φ is a symmetric piecewise linear continuous function

$$\varphi(\sigma) = \begin{cases} m_a\sigma - (m_b - m_a)\delta, & \sigma \leqslant -\delta, \\ m_b\sigma, & |\sigma| < \delta, \\ m_a\sigma + (m_b - m_a)\delta, & \sigma \geqslant \delta, \end{cases} \tag{1'}$$

with $m_a \neq m_b$, $\delta > 0$, *see Fig. 1. Assume that there exist* $\mu > 0$ *and* $\eta \in \mathbb{R}$, *such that the different linear parts satisfy*

$$\begin{aligned} \operatorname{Spec}(A + m_a b c^T) &= \{-\mu, -2\mu\}, \\ \operatorname{Spec}(A + m_b b c^T) &= \{\eta, 2\eta\} \end{aligned} \tag{1''}$$

and that the system is "observable", that is

$$\det\begin{pmatrix} c^T \\ c^T A \end{pmatrix} \neq 0.$$

Then the system (1) is topologically equivalent to the Liénard system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3(\alpha+1) \\ 2(\alpha^2-1) \end{pmatrix} \operatorname{sat}(x), \tag{2}$$

where $\alpha = \frac{\eta}{\mu}$ *and "sat" stands for the normalized saturation function*

$$\operatorname{sat}(x) = \begin{cases} -1, & \text{if } x < -1, \\ x, & \text{if } |x| \leqslant 1, \\ 1, & \text{if } x > 1. \end{cases}$$

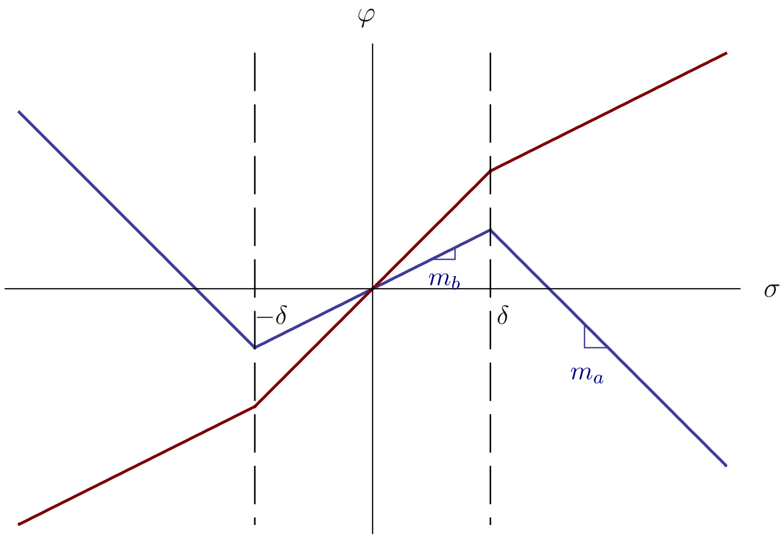

Fig. 1. Two typical cases for the symmetric piecewise linear function φ; in one of them the slopes m_a and m_b are indicated.