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Linearization of the products of the Carlitz–Srivastava polynomials of the first and second kinds via their integral representations

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ABSTRACT

In this paper, the authors investigate two families of generalized Lauricella polynomials which are known as the Carlitz–Srivastava polynomials of the first and second kinds. By means of their multiple integral representations, it is shown how one can linearize the product of two different members of each of these two families of the Carlitz–Srivastava polynomials. Upon suitable specialization of the main results presented in this paper, the corresponding integral representations are deduced for such familiar classes of multivariable hypergeometric polynomials as (for example) the Lauricella polynomials $F_D^{(r)}$ in r variables and the Appell polynomials F_1 in two variables. Each of these integral representations, which are derived as special cases of the main results in this paper, may also be viewed as a linearization relationship for the product of two different members of the associated family of multivariable hypergeometric polynomials.

1. Introduction and definitions

Over four decades ago, Srivastava [14] introduced and investigated the following general class of polynomials:

$$S_n^N(z) := \sum_{k=0}^{[n/N]} \frac{(-n)_{Nk}}{k!} \Lambda_{n,k} z^k \quad (N \in \mathbb{N} := \{1,2,3,\cdots\};\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

where $\{\Lambda_{m,n}\}_{m,n\in\mathbb{N}_0}$ is a suitably bounded double sequence of essentially arbitrary (real or complex) parameters, $[\kappa]$ denotes the greatest integer in $\kappa \in \mathbb{R}$ and $(\lambda)_\nu$ denotes the Pochhammer symbol or the *shifted factorial*, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0),$$

which is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists.

Srivastava's polynomials $S_n^N(z)$ and their variants have been considered, in recent years, by numerous other workers on the subject (see, for example, [5, p. 145 *et seq.*], [7], [8, p. 448 *et seq.*], [9,10] and [11,15,12]) (see also many other works

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on the polynomials belonging the general class $S_n^N(z)$, which are cited in each of these earlier references). On the other hand, motivated substantially by the work of Carlitz [2], Carlitz and Srivastava [3,4] considered two families of multivariable hypergeometric polynomials associated with the following particularly simple form of the (Srivastava–Daoust) generalized Lauricella function:

$$F^{1:1;\ldots;1}_{1:0;\ldots;0}\left(\begin{array}{l}[\alpha:\theta_1,\ldots,\theta_r]:[\beta_1:\phi_1];\ldots;[\beta_r:\phi_r];\\ \qquad\qquad\qquad\qquad\qquad\qquad x_1,\ldots,x_r\\ [\gamma:\psi_1,\ldots,\psi_r]:\quad —;\ldots;\quad —;\end{array}\right)$$
$$=\sum_{k_1,\ldots,k_r=0}^{\infty}\frac{(\alpha)_{k_1\theta_1+\cdots+k_r\theta_r}(\beta_1)_{k_1\phi_1}\cdots(\beta_r)_{k_r\phi_r}}{(\gamma)_{k_1\psi_1+\cdots+k_r\psi_r}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}$$
$$=: F_D^{(r)}\left[(\alpha:\theta_j):(\beta_j,\phi_j);(\gamma:\psi_j);x_1,\ldots,x_r\right], \tag{1}$$

where the parameters α,β_j and γ are arbitrary complex numbers and (for $j=1,\ldots,r$) the coefficients θ_j,ϕ_j and ψ_j are, in general, *nonnegative* real numbers.

Carlitz–Srivastava Polynomials of the First Kind (see [3,p. 43]):

$$F^{(r)}_{m,D}\left[(-m:m_j):(\alpha_j,\phi_j);(\gamma:\psi_j);x_1,\ldots,x_r\right]:=\sum_{k_1,\ldots,k_r=0}^{m_1k_1+\cdots+m_rk_r\leqq m}\frac{(-m)_{m_1k_1+\cdots+m_rk_r}(\alpha_1)_{\phi_1k_1}\cdots(\alpha_r)_{\phi_rk_r}}{(\gamma)_{\psi_1k_1+\cdots+\psi_rk_r}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}, \tag{2}$$

which obviously corresponds to (1) when

$$\alpha=-m,\quad \theta_j=m_j\quad\text{and}\quad \beta_j=\alpha_j\quad (j=1,\ldots,r;\ m\in\mathbb{N}_0;\ m_j\in\mathbb{N}).$$

Carlitz–Srivastava Polynomials of the Second Kind (see [4, p. 143]):

$$F^{(m_r)}_{m,D}\left[(\alpha:\mu_j):(-m_j,p_j);(\gamma:\nu_j);x_1,\ldots,x_r\right]:=\sum_{k_1=0}^{[m_1/p_1]}\cdots\sum_{k_r=0}^{[m_r/p_r]}\frac{(\alpha)_{\mu_1k_1+\cdots+\mu_rk_r}(-m_1)_{p_1k_1}\cdots(-m_r)_{p_rk_r}}{(\gamma)_{\nu_1k_1+\cdots+\nu_rk_r}}\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}, \tag{3}$$

which corresponds to (1) when

$$\theta_j=\mu_j,\quad \beta_j=-m_j\quad\text{and}\quad \phi_j=p_j\quad\text{and}\quad \psi_j=\nu_j\quad (j=1,\ldots,r;\quad m_j\in\mathbb{N}_0;\quad p_j\in\mathbb{N}).$$

Here, and in what follows, we find it to be convenient to abbreviate the array of r parameters $m_1,\ldots,m_r$ by (m_r) and we write

$$m_1+\cdots+m_r=m.$$

Indeed, in their special case when

$$\mu_j=\nu_j=p_j=1\quad (j=1,\ldots,r),$$

each of the Carlitz–Srivastava polynomials defined by (2) and (3) reduces to the Lauricella polynomials considered earlier in the aforementioned work by Carlitz [2, p. 270, Eq. (1.8)].

Our main objective in this investigation is first to derive several multiple integral representations associated with the multivariable hypergeometric polynomials defined by (2) and (3). We also consider several special cases and consequences of our main results. Each of the integral representations, which are derived in this paper, may be viewed also as a linearization relationship for the product of two different members of the associated family of multivariable hypergeometric polynomials.

2. Integral representations for the generalized Lauricella polynomials

First of all, by repeatedly applying such elementary series identities as follows (see, for example, [16, p. 52, Eq. 1.6(3)]):

$$\sum_{k_1,\ldots,k_r=0}^{\infty}f(k_1+\cdots+k_r)\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}=\sum_{k=0}^{\infty}f(k)\frac{(x_1+\cdots+x_r)^k}{k!}, \tag{4}$$

it is easily seen for a suitably bounded multiple sequence

$$\{\Omega(k_1,\ldots,k_r)\}_{k_1,\ldots,k_r\in\mathbb{N}_0}$$

that

$$\sum_{k_1,\ldots,k_r,\ell_1,\ldots,\ell_r=0}^{\infty}\Omega(k_1+\ell_1,\ldots,k_r+\ell_r)\frac{x_1^{k_1}}{k_1!}\cdots\frac{x_r^{k_r}}{k_r!}\frac{y_1^{\ell_1}}{\ell_1!}\cdots\frac{y_r^{\ell_r}}{\ell_r!}\quad=\sum_{k_1,\ldots,k_r=0}^{\infty}\Omega(k_1,\cdots,k_r)\frac{(x_1+y_1)^{k_1}}{k_1!}\cdots\frac{(x_r+y_r)^{k_r}}{k_r!} \tag{5}$$

provided that each of the series involved is absolutely convergent. In fact, upon setting

$$\Omega(k_1,\ldots,k_r)=f(k_1+\cdots+k_r)$$