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Stability of solutions of nonlinear neutral differential equations with piecewise constant delay and their discretizations ☆

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ABSTRACT

This paper is devoted to stability analysis of solutions of nonlinear neutral differential equations with piecewise constant delay. These results form the basis for obtaining insight into the analogous properties of numerical solutions generated by BN_f- stable Runge–Kutta methods. It is showed, theoretically and numerically, that two classes of Runge–Kutta methods considered in literature are contractive and asymptotically stable. Numerical experiments also demonstrate that a class of Runge–Kutta methods can preserve their original convergence order for ODEs but not the other class of Runge–Kutta methods.

1. Neutral delay differential equations with piecewise constant delay

The problems of interest are evolutionary problems of the type

$$y'(t) = f(t, y(t), y([t]), y'([t])), \quad t \geqslant 0, \tag{1.1a}$$

subject to

$$y(0) = \phi, \tag{1.1b}$$

where $[x]$ is the greatest integer less or equal to x, $\phi \in C^N$ is the initial datum, $f : [0,\infty) \times C^N \times C^N \times C^N \to C^N$ is continuous and satisfies the following conditions:

$$Re\langle y_1 - y_2, f(t, y_1, u, v) - f(t, y_2, u, v)\rangle \leqslant \alpha(t)\|y_1 - y_2\|^2, \quad \forall t \geqslant 0,\ y_1, y_2, u, v \in C^N, \tag{1.2a}$$

$$\|f(t, y, u, v_1) - f(t, y, u, v_2)\| \leqslant \gamma(t)\|v_1 - v_2\|, \quad \forall t \geqslant 0,\ y, u, v_1, v_2 \in C^N, \tag{1.2b}$$

$$\|H(t, y, u_1, v, w) - H(t, y, u_2, v, w)\| \leqslant \varrho(t)\|u_1 - u_2\|, \quad \forall t \geqslant 0,\ y, u_1, u_2, v, w \in C^N, \tag{1.2c}$$

where $\alpha(t)$, $\gamma(t)$, $\varrho(t)$ are continuous functions and

$$H(t, y, u, v, w) := f(t, y, u, f([t], u, v, w)).$$

This is a particular case of equations with piecewise continuous arguments, or EPCA (see the monograph of Wiener [19]), and is a particular neutral delay differential equations of the type

$$\begin{cases} y'(t) = f(t, y(t), y(\eta(t)), y'(\eta(t))), & t \geqslant 0, \\ y(t) = \phi(t), & t \leqslant 0, \end{cases} \tag{1.3}$$

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which provide a mathematical instrument to applied science [7,8,3,4].

In his book [19], Wiener presented some properties of the solution to the following linear problem

$$\begin{cases} y'(t) = A_0 y(t) + A_1 y([t]) + A_2 y'([t]), & t \geqslant 0, \\ y(0) = \phi, \end{cases} \tag{1.4}$$

where $A_0,\ A_1,\ A_2$ are constant $N \times N$ matrices and $y(t)$ is a vector of N elements.

Definition 1.1 (Wiener [19], see also [18]). A solution of (1.1) on $[0,\infty)$ is a function $y(t)$ that satisfies the conditions:

(i) $y(t)$ is continuous on $[0,\infty)$.
(ii) The derivative $y'(t)$ exists at each point $t \in [0,\infty)$, with the possible exception of the points $[t] \in [0,\infty)$ where one-sided derivatives exist.
(iii) Eq. (1.1) is satisfied on each interval $[m, m+1) \subset [0,\infty)$ with integral end-points. Throughout this paper, m denotes a non-negative integer.

Theorem 1.1 (*Wiener [19]*). *If the matrices A_0 and $B = I - A_2$ are nonsingular, then problem (1.4) has on $[0,\infty)$ a unique solution*

$$y(t) = E(\{t\})E^{[t]}(1)\phi, \tag{1.5}$$

where $\{t\}$ is the fractional part of t and

$$E(t) = I + (e^{A_0 t} - I)S \quad \text{and} \quad S = A_0^{-1} B^{-1}(A_0 + A_1).$$

In [11], Lv et al. investigated the stability of Euler–Maclaurin methods for more general linear neutral differential equations with piecewise continuous arguments. Following [18], Wang and Li in [13] gave the definition of the solution of problem (1.1) and considered the dissipativity of problem (1.1). In this paper, we concentrate on the stability of solutions to nonlinear problem (1.1). The uniqueness of (1.1) will be proved in Section 2. The solution of (1.1) may be written $y(t) \equiv y(\phi; t)$; then, the stability results are concerned with $y(\phi;t) - y(\psi;t)$ or, for numerical stability, approximations to $y(\phi;t) - y(\psi;t)$. The stability of the numerical solution produced by BN_f-stable Runge–Kutta methods (RKMs) is investigated in Section 3. Two numerical examples are given in the last section of this paper: one is to demonstrate the stability of the methods considered in this paper; the other is to illustrate the convergence of these methods.

As a special case of (1.1a) we have the delay differential equations (DDEs) with piecewise constant delay

$$y'(t) = f(t, y(t), y([t])), \quad t \geqslant 0, \tag{1.6}$$

which has been widely studied by applied mathematicians and numerical researchers [5,9,16,17,10]. The general theory and basic results for differential equations with piecewise constant delay have by now been thoroughly investigated in the book of Wiener [19].

2. The stability of the exact solutions

For any $\varphi \in C^N$ and any $t \geqslant 0$, we shall always assume that equations $x = f([t], \varphi, \varphi, x)$ has a unique bounded solution $x(\varphi; [t])$. Then we have the following stability results.

Theorem 2.1. *Suppose problem (1.1) satisfies conditions (1.2) and*

$$\alpha(t) < 0, \quad \gamma(t) < 1, \quad \frac{\varrho(t) - \alpha([t])\gamma(t)}{-\alpha(t)} \leqslant 1, \quad \forall t \geqslant 0. \tag{2.1}$$

Then we have

$$\|y(\phi;t) - y(\psi;t)\| \leqslant \|\phi - \psi\|, \quad \forall t \geqslant 0, \tag{2.2}$$

which means that this system is contractive.

Proof. To prove the theorem, let us define $Y(t) = \|y(\phi;t) - y(\psi;t)\|^2$. Then we have

$$\begin{aligned} Y'(t) &= 2Re\langle y(\phi;t) - y(\psi;t), y'(\phi;t) - y'(\psi;t)\rangle \\ &\leqslant 2\alpha(t)Y(t) + 2Re\langle y(\phi;t) - y(\psi;t), f(t, y(\psi;t), y(\phi;[t]), y'(\phi;[t])) - f(t, y(\psi;t), y(\psi;[t]), y'(\psi;[t]))\rangle \\ &\leqslant 2\alpha(t)Y(t) + 2\|y(\phi;t) - y(\psi;t)\|\Phi(t), \end{aligned} \tag{2.3}$$

where

$$\Phi(t) = \|f(t, y(\psi;t), y(\phi;[t]), y'(\phi;[t])) - f(t, y(\psi;t), y(\psi;[t]), y'(\psi;[t]))\|.$$