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Two-dimensional Riemann problem involving three J's for a hyperbolic system of nonlinear conservation laws ☆

Yicheng Pang [a], Jianjun Paul Tian [b], Hanchun Yang [a,*]

[a] Department of Mathematics, Yunnan University, Kunming 650091, PR China
[b] Department of Mathematics, College of William and Mary, Williamsburg, VA 23187, USA

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ABSTRACT

In this paper, we study the Riemann problem for a two-dimensional hyperbolic system of conservation laws with initial data projecting exactly three contact discontinuities. We use the generalized characteristic analysis method to analyze interactions of contact discontinuities, and obtain six analytical solutions and corresponding criteria. Delta shock waves and spiral structures appear in some solutions. We also conduct numerical experiments, and numerical results confirm the geometric structures of the constructed solutions.

1. Introduction

Although the hyperbolic systems of conservation laws are a fundamental principle in building mathematical models for many natural processes, it still lacks of a rigorous understanding. While the one-dimensional hyperbolic systems of conservation laws has been understood relatively well ([1–4] and the references therein), the analysis of multidimensional nonlinear hyperbolic systems of conservation laws remains a challenging problem. One of the essential difficulties in solving multidimensional systems of hyperbolic conservation laws stems from unclear relations between initial data and solutions. We do not have enough knowledge to connect initial data and the corresponding solutions. Riemann problems, where initial data possess the simplest geometries, such as step functions or step-like functions, are not only easy to study the existence, uniqueness and asymptotic behavior of solutions as $t \to \infty$ for corresponding Cauchy problems, but also provide "building block" for general initial data of Cauchy problems. And characteristic analysis or generalized characteristic analysis is suitable for Riemann problem. It is important to consider different sets of Riemann initial data for a given hyperbolic system of conservation laws, which may reveal some general properties of the system.

In this paper, we consider the following two-dimensional hyperbolic system

$$\begin{cases} u_t + (u^2)_x + (uv)_y = 0, \\ v_t + (uv)_x + (v^2)_y = 0. \end{cases} \tag{1.1}$$

The system (1.1) arises in several fields, such as elasticity theory, magnetohydrodynamics, and oil recovery process [5], where this system is used to model conservation laws in each specific situation. The system (1.1) also can be derived from the momentum conservation laws of two-dimensional compressible Euler equations by letting density and pressure be constants, and can be used to explain some elementary phenomena in gas dynamics, such as diffractions along wedges.

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* Corresponding author.
E-mail address: hyang@ynu.edu.cu (H. Yang).

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In earlier papers [6,7], Tan and Zhang investigated the Riemann problem for system (1.1) with the initial data in the four quadrants satisfying the following assumption:(H) *Outside a neighborhood* $N_0(N_0 \subset R^2 \times R_+)$ *of the origin, each jump in the initial data projects exactly one planar rarefaction wave, shock wave, or contact discontinuity.*

They classified the initial data into 23 cases and obtained 57 exact solutions and their corresponding criteria with the complicated geometric structures. Particularly, in [6], for the initial data containing only four contact discontinuities, they constructed eight exact solutions in which the spiral structures and delta shock waves appear to be components of the solutions. Yang and Zhang [8] confirmed the exact solutions presented in [6,7] numerically using the MmB scheme. For two constant initial data, Huang and Yang [9] relaxed the restrictive assumption (H), and constructed 16 exact solutions, in which delta shock waves appeared too. For the study of delta shock waves, we refer the reader to articles [6,10–17].

The system (1.1) has also been studied from other aspects by many authors. Wang [18] studied non-uniqueness of solutions to the system (1.1) under some special conditions. Lopes-Filho and Nussenzveig Lopes [19] studied the singularity formation. Chen [20] investigated how shock waves arise from smooth initial data. Chen et al. [21] analyzed the evolution of discontinuity and the formation of triple-shock pattern in solutions when initial discontinuity is a convex curve.

To understand how Riemann initial data effect the solutions of the system (1.1), we will focus on investigating three constant initial data. There are many studies on three-constant Riemann problem for other two-dimensional hyperbolic systems of conservation laws. For instance, for a single conservation law, Chen et al. [22] studied the dependence of the solution structures on the initial values as well as the angles of initial discontinuities. A special structure, called Guckenheimer structure, was discovered in [23–25]. For the zero-pressure gas dynamics consisting of three conservation laws involving the mass and momentum, Cheng et al. [26] obtained nine explicit solutions containing the Mach-reflection-like configurations. In fact, in a recent paper [27], we studied a case of three constant Riemann problem, where three-constant initial data separated by x-positive, y-positive and x-negative axes. The initial data were classified into eight different cases, and 13 exact solutions were obtained, where delta shock waves appeared again. In the present paper, we study the system (1.1) with general three constant initial data

$$(u, v)\|_{t=0} = \begin{cases} (u_1, v_1), & x > 0,\ y > 0, \\ (u_2, v_2), & x < 0,\ y > x\tan\alpha, \\ (u_3, v_3), & \text{otherwise}, \end{cases} \tag{1.2}$$

where $\alpha = \arctan(y/x),\ \alpha \in (\pi/2, \pi)\bigcup(\pi, 3\pi/2)$, is a polar angle, and $(u_i, v_i),\ i = 1, 2, 3$, are constant states satisfying (H). We will pay attention to the case where the initial data involve three contact discontinuities in this article. The other cases will be studied in the future.

According to the type of the contact discontinuity J, the initial data can be classified into two cases: (1) $J_{12}^- J_{23}^- J_{31}^-$, (2) $J_{12}^- J_{23}^+ J_{31}^+$. With the generalized characteristic analysis method, we study interactions of contact discontinuities, and obtain six exact solutions and their corresponding criteria. Contrasting with the same system which has four constant initial data separated by two axes with four contact discontinuities [6], we have a simple initial data classification and reduce numbers of exact solutions. However, it can still remain the mathematical substance of the problem and contain all the main geometric structures as exhibited in [6] including eight configurations. The solution structures are strikingly analogous each other. Especially, both delta shock waves and spiral structures appear in solutions. Moreover, it is even more difficult to construct exact solutions in our situation since the Riemann initial data are in general forms. We also emphasize that the two-dimensional Riemann problem with three-constant initial data for the 2×2 system (1.1) may fill a gap. Perhaps this is a very starting point for investigation of three-constant Riemann problem in multidimensional hyperbolic systems.

To confirm analysis of exact solutions we constructed, we conduct numerical experiments by employing the second-order nonoscillatory central scheme [28]. The numerical results agree closely with the geometric structures of the constructed analytical solutions.

The rest of the paper is organized as follows. In Section 2, we present some basic necessary knowledge as a preparation. In Section 3, we construct exact Riemann solutions. In Section 4, we demonstrate some typical numerical results to verify the constructed analytical solutions.

2. Preliminaries

2.1. Characteristic curves and planar elementary waves

We introduce some necessary knowledge about system (1.1), and present our classification of the initial data and some notations. For details, the reader is referred to [6,27].

Since both (1.1) and (1.2) are invariant under the self-similar transformation, we should seek the self-similar solutions $(u, v)(t, x, y) = (u, v)(\xi, \eta)$ $(\xi = x/t, \eta = y/t)$, for which (1.1) and (1.2) are transformed to the following boundary-value problem (b.v.p.) with boundary at infinity,

$$\begin{cases} -\xi u_\xi - \eta u_\eta + (u^2)_\xi + (uv)_\eta = 0, \\ -\xi v_\xi - \eta v_\eta + (uv)_\xi + (v^2)_\eta = 0, \end{cases} \tag{2.1}$$