Applied Mathematics and Computation 219 (2013) 4702–4709

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Properties of certain transforms defined by convolution of analytic functions

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ARTICLE INFO

Keywords:
Univalent
Starlike
Integral transform
Convolution

ABSTRACT

Let $\mathcal{A}$ be the class of all analytic functions f in the open unit disk $\mathbb{U}$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. For $\lambda > 0$, $\mathrm{Re}\, c > 0$ and $\alpha < 1$, two subclasses $\mathcal{P}(\lambda)$ and $\mathcal{S}_\alpha^*$ of $\mathcal{A}$ are introduced. In this paper, we find suitable conditions on λ, c and α such that for each $f \in \mathcal{P}(\lambda)$ satisfying $(z/f(z)) * F(1, c; c+1; z) \neq 0$ for all $z \in \mathbb{U}$, the function

$$G(z) = \frac{z}{(z/f(z)) * F(1, c; c+1; z)} \qquad (z \in \mathbb{U})$$

belongs to $\mathcal{P}(\lambda')$, $\mathcal{S}_\alpha^*$ or $\mathcal{S}^*(\alpha)$. Here $\mathcal{S}^*(\alpha)$ denotes the usual class of starlike of order α $(0 \leqslant \alpha < 1)$ in $\mathbb{U}$. We also determine necessary conditions so that $f \in \mathcal{P}(\lambda)$ implies that

$$\left| \frac{zG'(z)}{G(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta}, \quad |z| < r,$$

where $r = r(\lambda, c; \beta)$ will be specified.

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1. Introduction

Let $\mathcal{A}$ be the class of functions normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, we let $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{U}\}$. A function $f \in \mathcal{A}$ is said to be starlike if f is univalent and $f(\mathbb{U})$ is a starlike domain with respect to $z = 0$. The class of all starlike functions is denoted by $\mathcal{S}^*$. It is well known that $f \in \mathcal{A}$ is starlike with respect to the origin if and only if $f(0) = 0, f'(0) \neq 0$ and

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

For $\alpha < 1$, we define

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{U} \right\}$$

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http://dx.doi.org/10.1016/j.amc.2012.10.085

and

$$\mathcal{S}_{\alpha}^{*} = \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha, \quad z \in \mathbb{U} \right\}.$$

It is evident that $\mathcal{S}^{*}(0) \equiv \mathcal{S}^{*}$, and $\mathcal{S}_{\alpha}^{*} \subset \mathcal{S}^{*}(\alpha) \subset \mathcal{S}^{*}$ for $0 \leqslant \alpha < 1$.

For $\lambda > 0$, let $\mathcal{P}(\lambda)$ denote the class of all functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leqslant \lambda, \quad z \in \mathbb{U}.$$

Also, let $\mathcal{U}(\lambda)$ denote the class of all functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{2} - 1 \right| \leqslant \lambda, \quad z \in \mathbb{U}.$$

We remark that $f(z)/z \neq 0$ for $z \in \mathbb{U}$ for $f \in \mathcal{P}(\lambda)$ or $f \in \mathcal{U}(\lambda)$.

Obradović and Ponnusamy [5] proved that

$$\mathcal{P}(2\lambda) \subset \mathcal{U}(\lambda) \subset \mathcal{S} \quad \text{for } 0 \leqslant \lambda \leqslant 1.$$

Note that the function $g(z) = z + z^2/2$ belongs to $\mathcal{U}(1)$ but does not belong to the class $\mathcal{P}(2)$. We also know that $\mathcal{U}(1) \subsetneq \mathcal{S}$ (see [1,8]) and so, one has $\mathcal{U}(\lambda) \subsetneq \mathcal{S}$ for $0 \leqslant \lambda \leqslant 1$ which implies $\mathcal{P}(2\lambda) \subsetneq \mathcal{S}$ for $0 \leqslant \lambda \leqslant 1$. The inclusions $\mathcal{P}(2\lambda) \subset \mathcal{U}(\lambda) \subset \mathcal{S}$ improve the result of Nunokawa et al. [3] who proved that functions in $\mathcal{P}(2)$ are just univalent in $\mathbb{U}$. Obradović, Ponnusamy et al. discussed the functions $f \in \mathcal{U}(\lambda)$ in [4,5,7], given some conditions such that f belongs to some function class, for example, $\mathcal{S}^{*}$ or $\mathcal{S}^{*}(\alpha)$ et al. with missing coefficients. Further, the integral transform of various subclasses of $\mathcal{S}$ have been investigated by a number of authors, see for example the works of Ponnusamy [10], Ponnusamy et al. [11,12] and the references therein.

Recently, Obradović and Ponnusamy [6] found the conditions on λ and $c \in \mathbb{C}$ with $\mathrm{Re}\, c \geqslant 0 \neq c$ such that for each $f \in \mathcal{U}(\lambda)$ satisfying $\frac{z}{f(z)} * F(1, c; c+1; z) \neq 0$ for all $z \in \mathbb{U}$ the transform

$$G(z) = \frac{z}{(z/f(z)) * F(1, c; c+1; z)} \quad (z \in \mathbb{U})$$

is univalent or starlike. Here $*$ denotes the convolution (or Hadamard product) of analytic functions on $\mathbb{U}$:

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in \mathbb{U})$$

for

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

and $F(a, b; c; z)$ denotes the Gaussian hypergeometric function. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$, then,

$$zF(a, b; c; z) * f(z) := \sum_{n=1}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} a_n z^n \quad (c \neq -1, -2-3, \ldots; z \in \mathbb{U}),$$

where $(a)_n$ denotes the Pochhammer symbol defined by $(a)_0 = 1$ and $(a)_n := a(a+1) \cdots (a+n-1)$ for $n \in \mathbb{N}$.

The object of the present paper is to find conditions such that the function $G(z)$ is in $\mathcal{P}(\lambda')$, $\mathcal{S}_{\alpha}^{*}$ or $\mathcal{S}^{*}(\alpha)$ whenever $f \in \mathcal{P}(\lambda)$.

For the proof of our main results, we need the following lemma.

Lemma 1.1. *Suppose that $P(z) = p_2 z^2 + \cdots$ is analytic in the unit disk $\mathbb{U}$, $\lambda > 0$ and $c \in \mathbb{C}$ with $\mathrm{Re}\, c \geqslant 0 \neq c$ such that*

$$\left| P(z) + \frac{1}{c} z P'(z) \right| < \lambda, \quad z \in \mathbb{U}. \tag{1.2}$$

Then

$$|P(z)| < \lambda \frac{|c|}{|c+2|}, \quad z \in \mathbb{U}.$$

Proof. The proof is well-known and is an easy consequence of a result due to Hallenbeck and Ruscheweyh [2]. The same may be obtained also as the gap series version of Eq. (16) in [9] or as an easy consequence of a differential subordination result. □

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