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Conservation laws of the (2 + 1)-dimensional KP equation and Burgers equation with variable coefficients and cross terms

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ABSTRACT

In this paper, conservation laws for the (2 + 1)-dimensional KP equation and Burgers equation with variable coefficients and cross terms are studied. Due to the existence of cross terms, the "new conservation theorem" given by Ibragimov cannot be applied to the two equations directly. We propose two rules of modifications which ensure the theorem can be applied to nonlinear evolution equations with cross terms. Formulas of conservation laws for the KP equation and Burgers equation are given. Using these formulas, non-trivial and time-dependent conservation laws for these equations are derived according to their different Lie symmetries.

1. Introduction

The notion of conservation laws plays an important role in the study of nonlinear science. Construction of explicit forms of conservation laws is meaningful, as they are used for the development of appropriate numerical methods and for mathematical analysis, in particular, existence, uniqueness and stability analysis [1–3]. In addition, the existence of a large number of conservation laws of a partial differential equation (system) is a strong indication of its integrability. The famous Noether's theorem [4] provides a systematic way of determining conservation laws for Euler–Lagrange equations once their Noether symmetries are known, but this theorem relies on the availability of classical Lagrangians. To find conservation laws of differential equations without classical Lagrangians, researchers have made various generalizations of Noether's theorem [5–15]. Among those, the new conservation theorem given by Ibragimov [5] is one of the most frequently used methods.

For any differential equations (linear or nonlinear), the new conservation theorem offer a procedure for constructing explicit conservation laws associated with the known Lie, Lie-Bäcklund or non-local symmetries. Furthermore, it does not require the existence of classical Lagrangians. Using the conservation laws formulas given by the theorem, conservation laws for lots of equations have been studied [6–18]. When applying the new conservation theorem to a given nonlinear evolution equation with cross terms, we must be careful with the cross terms. If we apply the conservation laws formulas to equations with cross terms directly, it will result to errors. In this paper, take the (2 + 1)-dimensional KP equation and Burgers equation with variable coefficients and cross terms as examples, we give two rules of modifying the formulas of conservation laws. The correctness of the modifications is illustrated by the two equations.

The rest of the paper is organized as follows. In Section 2, we recall the main notations and theorems used in this paper. In Section 3, we first give formulas of conservation laws for the (2 + 1)-dimensional variable coefficient KP equation with cross terms. Using the formulas, conservation laws of the KP equation are derived. In Section 4, formulas of conservation laws for the (2 + 1)-dimensional variable coefficient Burgers equation with cross terms are proposed. Then

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we derive non-trivial conservation laws for the Burgers equation using the formulas. Some conclusions and discussions are given in the last section.

2. Preliminaries

In this section we briefly present the notations and theorems used in this paper. Consider a sth-order nonlinear evolution equation

$$F(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}) = 0, \tag{1}$$

with n independent variables $x = (x_1, x_2, \ldots, x_n)$ and a dependent variable u, where $u_{(1)}, u_{(2)}, \ldots, u_{(s)}$ denote the collection of all first, second, ..., sth-order partial derivatives. $u_i = D_i(u), u_{ij} = D_j D_i(u), \ldots$. Here

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots, \quad i = 1, 2, \ldots, n$$

is the total differential operator with respect to x_i.

Definition 1 [7]. The adjoint equation of Eq. (1) is defined by

$$F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}) = 0, \tag{2}$$

with

$$F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}) = \frac{\delta(vF)}{\delta u},$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{m=1}^{s} (-1)^m D_{i_1}, \ldots, D_{i_m} \frac{\partial}{\partial u_{i_1 i_2 \ldots i_m}}$$

denotes the Euler–Lagrange operator, v is a new dependent variable, $v = v(x)$.

Theorem 1 [5]. *The system consisting of Eq. (1) and its adjoint equation (2)*

$$\begin{cases} F(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}) = 0, \\ F^*(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}) = 0 \end{cases} \tag{3}$$

has a formal Lagrangian

$$L = vF(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}). \tag{4}$$

Theorem 2 [5]. *Consider the system (3) consisting of Eq. (1) and its adjoint equation (2). If Eq. (1) admits an operator*

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u}, \tag{5}$$

where X is either a Lie symmetry, i.e. $\xi^i = \xi^i(x, u), \eta = \eta(x, u)$, *or a Lie-Bäcklund operator, i.e.* $\xi^i = \xi^i(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(\ell)})$ *and* $\eta = \eta(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(j)})$ *are any differential functions, then Eq. (2) admits the operator (5) extended to the variable v by the formula*

$$X^* = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u} + \eta_* \frac{\partial}{\partial v},$$

where $\eta_* = -\left(\lambda v + v D_i(\xi^i)\right), \lambda$ *is determined by*

$$X(F) = \lambda F.$$

In the following we recall the "new conservation theorem" given by Ibragimov in [5].

Theorem 3. *Any Lie point, Lie-Bäcklund and non-local symmetry*

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u}$$

of Eq. (1) provides a conservation law $D_i(T^i) = 0$ *for the system comprising Eq. (1) and its adjoint equation (2). The conserved vector is given by*