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On the implementation of discontinuous Galerkin methods for Volterra integral equations with highly oscillatory Bessel kernels ☆

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ABSTRACT

In this paper, we consider implementation of discontinuous Galerkin methods for (weakly singular) Volterra integral equations of first and second kinds with highly oscillatory Bessel kernels, which cost the same operations independent of large values of frequencies. Preliminary numerical results show that the proposed method is efficient for verifying accuracy of approximate solutions.

1. Introduction

Many applications such as modeling radar, sonar, acoustic and electromagnetic scattering problems can be reformulated as integral equations with highly oscillatory Bessel kernels. For example, Davies and Duncan [5] studied the numerical solution of a scalar retarded potential integral equation posed on an infinite flat surface

$$\int_\Gamma \frac{u(x', t-|x'-x|)}{|x'-x|}dx' = a(x,t) \quad \text{on} \quad \Gamma\times(0,T),$$

which can be reformulated as a Volterra integral equation (VIE) of the first kind by taking continuous Fourier transforms

$$2\pi\int_0^x \hat{u}(\omega, x-t)J_0(rt)dt = \hat{a}(\omega,x), \quad \omega>0. \tag{1.1}$$

Eggermont [6] studied the convergence rate of Galerkin methods for Abel-type integral equation

$$\frac{1}{\Gamma(\alpha)}\int_0^x \frac{K(x,t)u(t)}{(x-t)^\alpha}dt = f(x), \quad 0<x<1, \quad 0\leqslant\alpha<1$$

and showed the convergence order for $u\in C^p[0,1]$ and $K(x,x)=1$

$$\|u-U\|_{L^\infty(0,1)} \leqslant c\inf\left\{\|u-\psi\|_{L^\infty(0,1)} : \psi\in\mathcal{S}_p^{-1}(I_{\Delta t})\right\}$$

for an approximate solution $U(x)$ from the space of discontinuous piecewise polynomials $\mathcal{S}_p^{-1}(I_{\Delta t})$.

Brunner et al. [4] introduced a discontinuous Galerkin (DG) method for

$$\int_0^x K(x-t)u(t)dt = f(x), \quad K(0)=1, \quad x\in[0,1], \tag{1.2}$$

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exhibited global convergence of order p when p is odd and order $p+1$ when p is even, and considered its application to (1.1).

These DG methods in [4,6] may suffer from difficulty for computation of Eq. (1.1) containing highly oscillatory kernels since the computation of the highly oscillatory integrals by standard quadrature methods is exceedingly difficult and the cost steeply increases with the frequency [9,10]. Particularly, for large values of ω, the length scale associated with the space mesh for DG methods is too small, which will induce a large scale and ill-conditioned linear system [3,4,14]. Furthermore, the DG method in [4] needs to behave stably under the assumption that $\omega \propto 1/\Delta t$. But VIE convergence analysis needs $\omega\Delta t \to 0$ as $\Delta t \to 0$.

This paper is to introduce implementation of DG methods to solve

$$\lambda u(x) + \int_0^x \frac{J_m(\omega(x-t))}{(x-t)^\alpha} u(t)dt = f(x), \quad x \in [0,1], \quad 0 \leqslant \alpha < 1, \quad \omega \gg 1, \tag{1.3}$$

where $u(x)$ is the unknown function whose value is to be determined in the interval $[0,1]$, $f(x)$ a given sufficiently smooth function, and J_m the Bessel function of the first kind of nonnegative order m. In particular, if $\lambda = 0$, the Eq. (1.3) is a (weakly singular) Volterra integral equation of first kind [2]. Numerical results in this paper show that the proposed method is efficient for verifying accuracy of approximate solutions, and the more larger of ω the more accuracy of the approximation.

2. Discontinuous Galerkin method for (1.3)

We restrict our attention to a uniform grid with spacing $\Delta t = 1/N$ for some integer $N > 0$. Set $t_n = n\Delta t$ for $n = 0:N$ and denote the set of grid points by

$$I_{\Delta t} = \{t_n : \ 0 \leqslant n \leqslant N\}.$$

Following the notation of [2,4], an approximate solution $U(x)$ of (1.3) is from the space of discontinuous piecewise polynomials

$$S_p^{(-1)}(I_{\Delta t}) = \left\{ v \in L^2(0,1) : \ v|_{I_n} \in \mathcal{P}^{(p)}(I_n), \quad n = 0:N-1 \right\},$$

where $\mathcal{P}^{(p)}(I_n)$ denotes the space of all real polynomials of degree not exceeding p on the interval $I_n = (t_n, t_{n+1}]$. It is convenient to represent the approximate on each subinterval as

$$U(t_n + s\Delta t) = \sum_{k=0}^{p} U_k^n s^k \quad \text{for } s \in (0,1], \quad (U^n)_k = U_k^n, \quad k = 0:p, \quad n = 0:N-1.$$

The DG method for (1.3): Rewrite $U(t)$ as

$$U(t) = \sum_{n=0}^{N-1}\sum_{k=0}^{p} U_k^n \phi_k^n(t), \quad \phi_k^n(t) = \phi_k^n(t_n + s\Delta t) = \begin{cases} s^k & \text{if } s \in (0,1], \\ 0 & \text{otherwise} \end{cases}$$

for $k = 0:p$ and $n = 0:N-1$, which, together with (1.3), gives

$$(\phi_j^n, f) = \sum_{\ell=0}^{N-1}\sum_{k=0}^{p}\left[\left(\phi_j^n, \int_0^t \frac{J_m(\omega(t-w))}{(t-w)^\alpha}\phi_k^\ell dw\right)U_k^\ell + \lambda(\phi_j^n, \phi_k^\ell)U_k^\ell\right]$$

for each $j = 0:p$, where $(\cdot,\cdot)$ denotes the L^2-inner product on $[0,1]$.

The DG method is defined similar to [4] as follows

$$\left(\frac{\lambda}{\Delta t}C + B^0\right)U^n = a^n - \sum_{k=0}^{n-1} B^{n-k}U^k, \quad n = 0:N-1, \tag{2.1}$$

where $C = (c_{j,k}) = \frac{1}{j+k+1}$, $a_j^n = \frac{1}{\Delta t}\int_0^1 s^j f(t_n + s\Delta t)ds$, $B^n = \frac{1}{\Delta t^\alpha}(\beta_{j,k}^n)$, and $\beta_{j,k}^n$ for $j,k = 0:p$ are defined by

$$\beta_{j,k}^n = \begin{cases} \int_0^1 s^j \int_0^s \frac{J_m(\omega\Delta t s')}{s'^\alpha}(s-s')^k ds' ds, & n = 0, \\ \int_0^1 s^j \int_{n-1+s}^{n+s} \frac{J_m(\omega\Delta t s')}{s'^\alpha}(n+s-s')^k ds' ds, & n > 0 \end{cases} \tag{2.2}$$

and the sum on the right-hand side in (2.1) is defined to be zero when $n = 0$. For more detail, see [4].

The computation of a_j^n and $\beta_{j,k}^n$: a_j^n can be efficiently evaluated by the Gauss–Legendre quadrature Q_M^{GL}

$$a_j^n \approx \frac{1}{2\Delta t}\sum_{i=0}^{M} w_i \left(\frac{1+x_i}{2}\right)^j a\left(t_n + \frac{1+x_i}{2}\Delta t\right),$$

where $\{x_j\}_{j=0}^M$ are the Gauss nodes in $[-1,1]$ and $\{w_j\}_{j=0}^M$ are the corresponding weights, which can be efficiently computed by Glaser et al. [7]. The MATLAB file could be found from CHEBFUN [13]. In this paper, we set $M = 100$.