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# Unsteady flows of a class of novel generalizations of the Navier–Stokes fluid

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ABSTRACT

In this short paper we study the counterpart, within the context of a general class of fluids, of two famous unsteady flows originally studied by Stokes, within the context of Navier–Stokes fluid, namely Stokes' first and second problems. The class of fluids considered, stress power-law fluids, are capable of stress thinning or stress thickening and can describe phenomena that the classical power-law fluids are incapable of modeling. Within the context of the problems considered, we are able to find solutions wherein stress boundary layers develop.



## 1. Introduction

In his famous paper on the effect of internal friction on the motion of pendulums, Stokes [1] studied two interesting unsteady flow problems of a Navier–Stokes fluid in addition to several other issues, namely the flow due to an infinite accelerating plate and an infinite oscillating plate, on the fluid above it; these two problems are referred to as Stokes, first and second problem. The flow due to an infinite oscillating plate was also studied by Rayleigh [2]. The problems studied by Stokes were in an unbounded domain and by using a similarity transformation he was able to reduce the equation to a linear equation in two variables and he was able to obtain an explicit exact solutions. The early study of Stokes has been repeated within the context of a variety of non-Newtonian fluids and the studies are so numerous that we shall not document any of them. Suffice it is to say that as the flow has relevance to several interesting practical problems, in an idealized sense, such studies are warranted. In this paper, we study the counterparts to Stokes, first and second problems as well as other related initial-boundary value problems within the context of a new class of fluid models that was proposed recently (see [3]). We are interested in a class of fluid models, which could be considered as generalizations of the Navier–Stokes fluids; however there are yet markedly different from the usual non-Newtonian generalizations such as the classical power-law fluids or the more general Stokesian fluids (see [4]) and thus a few comments concerning the origins and the rationale for the models are warranted. We shall provide a short justification for the development of such models in the following remarks.

In the classical Navier–Stokes fluid model and its usual non-linear generalizations such as the power-law fluids (see [5]) or generalized Stokesian fluids (see [4]) or fluids of the differential type of grade $n$, $n > 1$ (see [6]), or for that matter very general fluids with memory such as Simple fluids (see [7]), constitutive specifications are postulated for the Cauchy stress in terms of kinematical quantities such as the velocity gradient or the history of kinematical quantities such as the history of the relative deformation gradient.

Constitutive specification for the stress in terms of the kinematics in clearly not in keeping with the notion of causality that is at the heart of Newtonian mechanics. Whether it was Newton or Hooke, or more recent authorities in continuum

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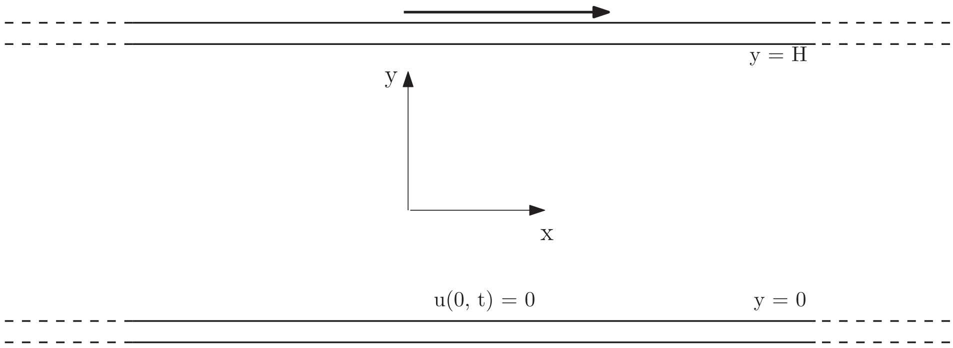


**Fig. 1.** Illustration of the flow domain.

mechanics, force is very clearly identified as the cause and kinematical measures, whether it is acceleration in the case of a particle or a rigid body, or the strain or the velocity gradient in a continua, are considered as effects. It might not be possible to describe the effect explicitly in terms of the cause, that is, to provide an expression for an appropriate kinematical quantity in terms of the stress; it might only be possible to provide a relation between the stress and the relevant kinematical quantities. However a procedure that provides an expression for the cause in terms of the effect seems somewhat peculiar, to say the least. For instance, within the context of continuum mechanics, Truesdell [8] remarks *"A constitutive equation is a relation between forces and motions. In popular terms, force is applied to a body to "cause" it to undergo a motion and the motion "caused" differs according to the nature of the body. In continuum mechanics the forces of interest are contact forces, which are specified by the stress tensor* **T**" Thus, it is rather surprising that the same author, as well as most others, specify expressions for the stress, the cause, in terms of the kinematics. This is most probably due to the simplification that results in the balance of linear momentum reducing to an equation for the displacement or the velocity field instead of having to deal with a very large system of non-linear partial differential equations, the balance equations and the constitutive equations simultaneously. (see [9] for a detailed discussion of this issue).

Recently, in a marked departure from such a specification for constitutive equations, on the basis of causality that is central to Newtonian mechanics, Rajagopal [10–12] articulated the need for constitutive specifications wherein an expression is provided for an appropriate kinematical quantity in terms of the stress or its history, if at all possible. However, as mentioned earlier it might not be possible to provide explicit expression for the kinematical quantity in terms of the stress, in which case we might need to provide fully implicit equations that relate the kinematical quantity/quantities (possibly their histories) and the history of the stress (see [10–12]). The recent paper on implicit constitutive relations for materials with fading memory (see [13]) offers a constitutive framework, which within the context of retarded motion delivers both fluids of the differential type such as the fluids of grade n (which includes the classical Navier–Stokes model) as well as fluids of the rate type like those due to Maxwell and Oldroyd (see [14,15]).

All the flows that are considered in this paper are unsteady flows that take place in the domain between two infinite parallel plates (Fig. 1), the bottom plate always being held fixed while the top plate is subject to a variety of initial and boundary conditions. Two of the problems that we consider correspond to the top plate being subject to a sudden velocity and a sinusoidal oscillation, respectively. Unlike the problems considered by Stokes, the flow under consideration takes place in a bounded domain. In the case of an unbounded domain, we would have to truncate the domain and solve the finite domain problem and then let the distance between the plates to increase to determine whether the solution tends to a limit as the distance tends to infinity. It is also worth observing that unlike the model considered by Stokes, namely the Navier–Stokes fluid model, which is a linear model, which due to the assumed form for the velocity field leads to a linear equation; we have to confront a highly non-linear system of partial differential equations which is not amenable to obtaining an exact solution. Hence we have to resort to using a numerical technique and we use the finite element method.

The arrangement of the paper is as follows. In Section 2, we introduce the new class of models. This is followed in the next section by the development of the governing equations for the problems under consideration. In Section 4, we very briefly introduce the finite element method and in Section 5, we study counterparts to Stokes, first problem; in Section 5.1 we consider the top plate, which is initially at rest, to be suddenly given a non-zero velocity at time $t = 0$, while in Section 5.2 we allow the top plate to be moved due to a suddenly applied shear stress at time $t = 0$. There is a reason for considering the problem of the top plate being sheared suddenly at time $t = 0$. In the problem that we consider, the constitutive relation is assumed for the symmetric part of the velocity gradient in terms of the stress, as we want to reflect the fact that the stress causes the motion. It would then be much more in keeping with the philosophy of the problem to ask the question, how does the fluid respond when the traction/stress (the cause) is suddenly applied. In Section 6, we study the counterparts to Stokes, second problem; in Section 6.1 the velocity of the top plate is assumed to vary sinusoidally while in Section 6.2 the shear stress applied to the top plate is assumed to oscillate sinusoidally. In the final section, we provide some concluding remarks concerning the problems that we have studied.