Applied Mathematics and Computation 219 (2013) 10053–10058

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Koçak's method shows that an unwittingly exaggerated convergence order is in fact 2

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ARTICLE INFO

*Keywords:*
Non-linear equations
Iterative methods
Newton's method
Convergence acceleration

ABSTRACT

Iterative solution of a nonlinear equation $f(x) = 0$ usually means a repetitive scheme to locate a fixed point of a related equation $x = g(x)$. Koçak's acceleration method smoothly gears up iterations with the aid of a superior secondary solver $g_K = x + G(x)(g(x) - x) = (g(x) - m(x)x)/(1 - m(x))$ where $G(x) = 1/(1 - m(x))$ is a gain and $m(x) = 1 - 1/G(x)$ is a straight line slope. The accelerator shows that a previously published article [A. Biazar, A. Amirtemoori, An improvement to the fixed point iterative method, AMC 182 (2006) 567–571] unwittingly exaggerated the convergence order of the solver it presented. This solver boils down to an indirect application of Newton's method solving $g(x) - x = 0$ which means that it is of second order. Hence, their claim that it "*increases the order of convergence as much as desired*" is false! The scheme wastes higher derivatives.



## 1. Introduction

A nonlinear equation

$$x = g(x) \tag{1}$$

can be solved by a repetitive scheme $x_{k+1} = g(x_k)$ where $k$ is the iteration count. If $z$ satisfies (1), then $z$ is called a *fixed-point* of $g$. Let $\varepsilon_k = x_k - z$. If there exist a real number $n$ and nonzero constant $c$ such that

$$\lim_{k\to\infty} (|\varepsilon_{k+1}|/|\varepsilon_k|^n),$$

then $n$ and $c$ are respectively called the *convergence order* and the *asymptotic error constant*. According to Traub [5], if $n$ is integral, then

$$c = \lim_{k\to\infty} \left(\varepsilon_{k+1}/\varepsilon_k^n\right) = g^{(n)}(z)/n!$$

If $n = 1$, then $g'(z) \neq 0$. An integral convergence order $n > 1$ means that

$$g'(z) = g''(z) = \cdots = g^{(n-1)}(z) = 0, \quad g^{(n)}(z) \neq 0. \tag{2}$$

$\varepsilon_{k+1}$ is proportional to $\varepsilon_k^n$ in the vicinity of $z$. Thus, $n$th order solvers are a subset of $(n-1)$th order solvers. Past this point, a function mentioned without an argument implies that the latter is $x$. Note also that a subscript starting with *a capital letter* is assigned to a solver $g$, convergence order $n$, and asymptotic error constant $c$ to indicate the person to whom $g$ is attributed.

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0096-3003/$ - see front matter 
http://dx.doi.org/10.1016/j.amc.2013.03.034

**Nomenclature**

| | |
|---|---|
| $c$ | asymptotic error constant |
| $c_K$ | asymptotic error constant of Koçak's solver |
| $f$ | nonlinear function to be solved |
| $g$ | nonlinear solver |
| $g_{id}$ | ideal solver, $g_{id} = z$ |
| $g_{pl}$ | piecewise linearisation |
| $g_{ps}$ | partial substitution |
| $g_K$ | Koçak's solver (accelerator) |
| $g_N$ | Newton's second-order solver |
| $G$ | partial substitution gain |
| $k$ | iteration counter |
| $m$ | straight line slope |
| $m_{id}$ | ideal linearisation slope |
| $n$ | convergence order |
| $n_K$ | convergence order of Koçak's solver |
| $x$ | independent variable |
| $w$ | weight (see Eq. (5)) |
| $w_{id}$ | limit for $w$ at $z$ (determined by $n$) |
| $z$ | fixed-point |
| $\varepsilon_k$ | error at the $k$th iteration, $\varepsilon_k = x_k - z$ |

## 2. Koçak's accelerator

This method [2–4] accelerates a given $g$ by generating and solving a secondary solver $g_K$ through the fixed-point preserving transformation

$$g_K = x + G(g - x) = (g - mx)/(1 - m), \quad G = 1/(1 - m), \quad m = 1 - 1/G, \tag{3}$$

where $G$ is a gain and $m$ is a straight line slope. Thus, $g_K$ is both partial substitution ($g_{ps}$) and piecewise linearisation ($g_{pl}$) that is $g_K \equiv g_{ps} \equiv g_{pl}$. (A straight line of slope $m$ joins $(x,g)$ to $(g_{pl},g_{pl})$.) The second version of Koçak's accelerator [3,4] has three cases:

(a) If $n = 1$, then $n_K = 3$, $c_K = g_K'''(z)/3!$, and $g_K'''(z) = -0.5g'''(z)/(1 - g'(z))$.
(b) If $n = 2$, then $n_K = 3$, $c_K = g_K'''(z)/3!$, and $g_K'''(z) = -0.5g'''(z)$.
(c) If $n > 2$, then $n_K = n + 1$, $c_K = g_K^{(n+1)}(z)/(n + 1)!$, and $g_K^{(n+1)}(z) = -g^{(n+1)}(z)/n$.

**Remark 1.** The '*ideal*' solver is $g_{id} = z$ which needs just one trial from *any* starting point. Albeit, $z$ is unknown until the end! $g_{id}$ can be harnessed in post priori analysis, comparative studies, and troubleshooting. *Regardless of the multiplicity of z* [2], the ideal slope is $m_{id} = (g_k - z)/(x_k - z) = \varepsilon_{k+1}/\varepsilon_k$. On the other hand, according to the mean value theorem for derivatives,

$$(g - z)/(x - z) = (g - g(z))/(x - z) = g'(x_i), x_i \in (x, z).$$

(Note that this does not necessarily mean $g'(x_i) \in (x,z)$!)

In line with (3), the set of requirements for $n_K$ to exceed 1, is $g_K^{(i)}(z) = 0,\ i = 1, 2, \ldots, n_K - 1$. It has been previously shown [2–4] that an equivalent set of *end-point* conditions is

$$m^{(i-1)}(z) = g^{(i)}(z)/i, \quad i = 1, 2, \ldots, n_K - 1, \quad m^{(0)}(z) = m(z) = g'(z). \tag{4}$$

There is plenty room to tune $m$ subject to (4).

**Remark 2.** If $m = g'(z)$, then $m(z) = g'(z)$ and so $n_K = 2$.

**Remark 3.** If $m = g'$, then $m'(z) = g''(z)$, $m''(z) = g'''(z)$, and so on. In the light of (4), $n_K = 2$ only because $m'(z) \neq g''(z)/2$ unless $g''(z) = 0$. The application is equivalent [2] in this case to utilising Newton's method ($g_N$) solving a secondary function $g - x = 0$. Indeed,

$$g_N = x - (g - x)/(g' - 1) = (g - mx)/(1 - m), \quad m = g'.$$

**Remark 4.** If $m = (g' + g'(z))/2$, then $m(z) = g'(z)$, $m'(z) = g''(z)/2$, $m''(z) = g'''(z)/2$, and so on. As a result, $n_K = 3$ since $m''(z) \neq g'''(z)/3$ unless $g'''(z) = 0$.

**Remark 5.** Let

$$m = wg' + (1 - w)g'(z) = g'(z) + w(g' - g'(z)), \tag{5}$$