Applied Mathematics and Computation 219 (2013) 7286–7295

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Three anisotropic benchmark problems for adaptive finite element methods

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ARTICLE INFO

Keywords:
Benchmark problem
Anisotropic solution
Boundary layer
Finite element method

ABSTRACT

In this paper we provide three benchmark problems with known exact solutions that can be used to assess the ability of adaptive finite element algorithms to handle anisotropically-behaved solutions. The first one is a Poisson equation with a smooth solution that only changes in one spatial direction. The second one is a singularly-perturbed linear elliptic equation whose solution exhibits a boundary layer, and the last one is a two-equation system that contains a boundary layer in one solution component only. In an appendix we show sample results obtained with the open source library HERMES. (http://hpfem.org/hermes.)

1. Introduction

The number of adaptive finite element libraries is growing – let us mention (in alphabetical order) for example Alberta [9], DealII [10], FEniCS [11], FETK [12], Hermes [13], libMesh [14], Phaml [15], PHG [16], 2dhp90 [17] and there are many others. A natural question that arises is how do they compare to each other? Unfortunately, comparison efforts are usually inhibited at the very beginning by diverse installation requirements, supporting libraries, input and output data formats, and different usage of various codes. And even if these problems could be overcome, not many benchmarks with known exact solutions are available to test various aspects of automatic adaptivity.

At this point we would like to acknowledge the pioneering work of Dr. William Mitchell (NIST) who not only collected a suite of 12 benchmark problems for adaptive FEM [3], but who also implemented and compared several *hp*-adaptive finite element algorithms by various authors [4].

This paper presents three benchmark problems with anisotropically-behaved solutions (not contained in [3]) that are designed to test the ability of adaptive algorithms to handle anisotropic refinements. The benchmark problems and their solutions are formulated in Sections 2–4. In a separate second part of the paper (Appendix A) we show for illustration sample results obtained with the open source HERMES library (http://hpfem.org/hermes).

2. Benchmark No. 1 "beginner"

The first benchmark problem is a Poisson equation

$$-\Delta u = \sin(x), \tag{1}$$

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2010.12.080

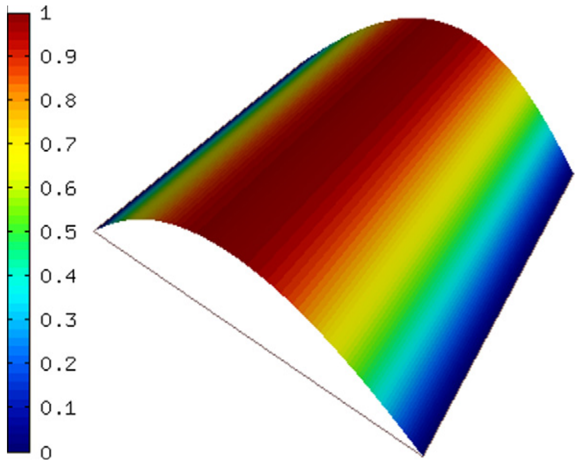

Fig. 1. Exact solution to the first benchmark problem.

in the domain $\Omega = (0,\pi)^2$, equipped with a zero Dirichlet boundary condition on the left edge, zero Neumann boundary conditions on the top and bottom edges, and a Neumann condition $\partial u/\partial \nu = -1$ on the right edge. Here ν is the unit outer normal vector to the boundary. The exact solution $u(x,y) = \sin(x)$ to this problem is shown in Fig. 1.

The goal of the benchmark is to attain a relative error below 10^{-4}% in the H^1-norm with as few degrees of freedom (DOF) as possible. Using 1D analysis [8], one can show that the minimum number of DOF needed is 16.

3. Benchmark No. 2 "intermediate"

Next we consider a singularly perturbed elliptic equation

$$-\Delta u + k^2 u - k^2 = g, \tag{2}$$

where k is a real constant. Here we use the value $k = 100$ and the problem can be made harder when k is increased. Eq. (2) is equipped with homogeneous Dirichlet boundary conditions and solved in the domain $\Omega = (-1,1)^2$. For convergence studies we use a manufactured exact solution

$$v(x,y) = \hat{u}(x)\hat{u}(y), \tag{3}$$

shown in Fig. 2, where

$$\hat{u}(x) = 1 - \frac{e^{kx} + e^{-kx}}{e^k + e^{-k}} \tag{4}$$

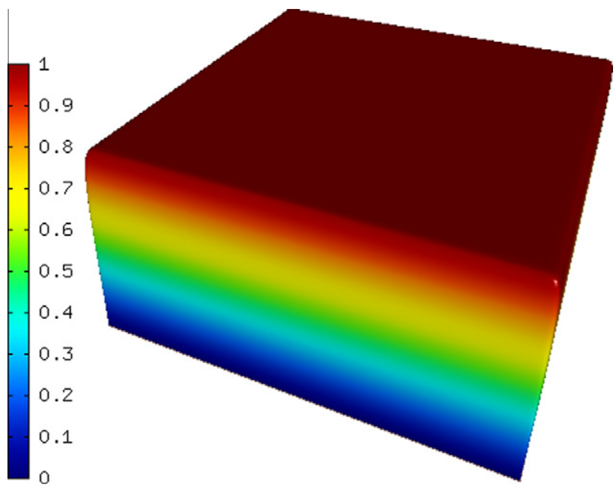

Fig. 2. Exact solution to the second benchmark problem for $k = 100$.