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Application of compact finite-difference schemes to simulations of stably stratified fluid flows

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ABSTRACT

This paper presents a comparison of the results of numerical simulations obtained by two different numerical methods for one specific case of stably stratified incompressible flow. The focus in this paper is on the numerical results obtained using some of the compact finite-difference discretizations introduced in the paper [1]. The numerical scheme itself follows the principle of semi-discretisation, with high order compact discretisation in space, while the time integration is carried out by the Strong Stability Preserving Runge–Kutta scheme. Results are compared against the reference solution obtained by the AUSM finite volume method. The test case used to demonstrate the capabilities of selected numerical methods represents the flow of stably stratified fluid over low, smooth, hill-like wall mounted obstacle.

1. Introduction

The mathematical modelling and numerical simulation of stratified fluid flow presents a challenging problem with many interesting applications. Stably stratified flows form a subclass of variable density flows where the mean flow density gradient points in the direction gravity force. Such a specific configuration results in a buoyancy force that is responsible for very distinct flow behaviour manifested by a presence of large-scale wave patterns in the flow-field.

From the numerical point of view, the simulations of stratified fluid flows are in general more demanding than the solution of similar non-stratified flow cases (see our previous work [2–5] or [6]). There are several arguments supporting this statement. First of all, the governing system describing the variable density fluid flow involves one more equation with respect to classical (constant density) incompressible Navier–Stokes equations. It is the transport equation for the density (or its perturbation), which is coupled to momentum equations by a buoyancy term (see Section 2). As a consequence of this buoyant force the obstacles in flow generate waves that propagate at long distances. These waves need to be properly resolved, without unphysical damping or dispersion. Moreover the appearance of these waves should not be significantly affected by the artificial boundaries of the computational domain. All these aspects of stably stratified fluid flows are addressed in this paper.

The above mentioned characteristic behaviour led us to the idea to use some of the compact finite-difference schemes for the simulation of stratified fluid flows. These schemes have successfully been applied in the field of computational aeroacoustics where they became a standard tool in simulations of wave propagation phenomena. These methods offer a relatively simple way of construction of formally high order discretisations with well defined diffusion and dispersion properties.

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2. Mathematical model

This section describes in detail the development of the simplified mathematical model for variable density incompressible flow. The emphasise is on the explanation of the simplifications made in the model and their justification.

2.1. Full incompressible model

The motion equations describing the flow of incompressible fluid could be written in the following general form [1]:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0, \tag{1}$$

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) = \operatorname{div}\mathbb{S} + \rho \boldsymbol{f}. \tag{2}$$

These equations should be complemented by the incompressibility constraint div$\boldsymbol{u}$ = 0 and the rheological constitutive relation for Newtonian incompressible fluid $\mathbb{S} = -p\mathbb{I} + 2\mu\mathbb{D}$ (with $\mathbb{D}$ being the symmetric part of the velocity gradient and $\mathbb{I}$ standing for the identity tensor). Expressing further the gravity force vector as $\boldsymbol{f} = (0,0,g)$, leads (exactly) to the following set of governing equations for unknowns $\boldsymbol{u}$, p and ρ:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{4}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho u v)}{\partial y} + \frac{\partial(\rho u w)}{\partial z} = -\frac{\partial p}{\partial x} + \mu \Delta u, \tag{5}$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho u v)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho v w)}{\partial z} = -\frac{\partial p}{\partial y} + \mu \Delta v, \tag{6}$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho u w)}{\partial x} + \frac{\partial(\rho v w)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} = -\frac{\partial p}{\partial z} + \mu \Delta w + \rho g. \tag{7}$$

The conservation of mass is represented by the continuity Eq. (4) can be written in the vector form

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0. \tag{8}$$

Using the chain rule of differentiation this equation can be rewritten as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \operatorname{grad} \rho = -\rho \operatorname{div} \boldsymbol{u}. \tag{9}$$

For incompressible flows satisfying the divergence-free constraint div $\boldsymbol{u}$ = 0, the right-hand side of Eq. (9) vanishes and thus the continuity equation reduces to transport (advection) equation for density ρ:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \operatorname{grad} \rho = 0. \tag{10}$$

This equation can be used together with the chain rule of differentiation to rewrite the left-hand side of the momentum balance (2):

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) = \boldsymbol{u}\frac{\partial \rho}{\partial t} + \rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{u}(\boldsymbol{u} \cdot \operatorname{grad} \rho) = \rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u})\right) + \boldsymbol{u}\underbrace{\left(\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \operatorname{grad} \rho\right)}_{=0}. \tag{11}$$

Using the above described manipulation, the governing system (3)–(7) could alternatively be rewritten as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{12}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{13}$$

$$\rho\left(\frac{\partial u}{\partial t} + \frac{\partial(u^2)}{\partial x} + \frac{\partial(u v)}{\partial y} + \frac{\partial(u w)}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu \Delta u, \tag{14}$$

$$\rho\left(\frac{\partial v}{\partial t} + \frac{\partial(u v)}{\partial x} + \frac{\partial(v^2)}{\partial y} + \frac{\partial(v w)}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu \Delta v, \tag{15}$$

$$\rho\left(\frac{\partial w}{\partial t} + \frac{\partial(u w)}{\partial x} + \frac{\partial(v w)}{\partial y} + \frac{\partial(w^2)}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu \Delta w + \rho g. \tag{16}$$

[1] Here we assume that the energy equation is decoupled from the equations of motion and the consequences of this simplification could be neglected. This e.g. means that heat production due to the mechanical energy dissipation is neglected.