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Symmetric positive solutions of boundary value problems with integral boundary conditions ☆

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ABSTRACT

Under conditions weaker than those used by Ma, we establish various results on the existence and nonexistence of symmetric positive solutions to fourth-order boundary value problems with integral boundary conditions. The arguments are based upon a specially constructed cone and the fixed point theory in a cone. Our results improve and extend that obtained in [H. Ma, Symmetric positive solutions for nonlocal boundary value problems of fourth order, Nonlinear Anal. 68 (2008) 645–651]. We illustrate our results by one example, which cannot be handled using the existing results.

1. Introduction

Boundary value problems for ordinary differential equations arises in different areas of applied mathematics and physics and so on, the existence and multiplicity of positive solutions for such problems have become an important area of investigation in recent years. To identify a few, we refer the reader to [1–8,10,14–34,61–65] and references therein. In particular, we would like to mention some results of Anderson and Avery [3], Graef et al. [4], and Yao [5]. In [12], Anderson and Avery studied the following fourth-order four point right focal boundary value problem

$$\begin{cases} x^{(4)}(t) = f(x(t)), \quad t \in [0,1], \\ x(0) = x'(q) = x''(r) = x'''(1) = 0, \end{cases}$$

where $0 < q < r < 1$ is two constants, $f : R \to R$ is continuous. By employing the five functionals fixed-point theorem, the authors gave sufficient conditions for the existence of three positive solutions of the above problem.

In [4], Graef, Qian and Yang considered the following fourth-order three point boundary value problem

$$\begin{cases} x^{(4)}(t) = \lambda g(t) f(x(t)), \quad 0 < t < 1, \\ x(0) = x'(1) = x''(0) = x''(p) - x''(1) = 0, \end{cases}$$

where $p \in (0,1)$ is a constant. The authors obtained the existence and nonexistence of positive solution by using the Krasnoselskii's fixed point theorem.

In [5], by using Guo-Krasnosel'skii fixed point theorem of cone expansion–compression type, Yao established several local existence theorems concerned with n positive solutions for the following fourth-order two point boundary value problem

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$$\begin{cases} x^{(4)}(t) = f(t, x(t)), & 0 \leqslant t \leqslant 1, \\ x(0) = x(1) = x'(0) = x''(1) = 0. \end{cases}$$

We notice that a type of symmetric problem is received much attention. For example, in [6], Yao considered the following two-point boundary value problem

$$\begin{cases} x''(t) + w(t)f(x(t)) = 0, & t \in (0,1), \\ \alpha x(0) - \beta x'(0) = 0, & \alpha x(1) + \beta x'(1) = 0. \end{cases}$$

The author obtained the existence of n symmetric positive solutions and established a corresponding iterative scheme by using monotone iterative technique.

In [7], Avery and Henderson studied the existence of multiple symmetric positive solutions for the following two-point boundary value problem

$$\begin{cases} x''(t) + f(x(t)) = 0, & t \in [0,1], \\ x(0) = 0 = x(1). \end{cases}$$

The main tool is a fixed point theorem due to Avery [8] which is a generalization of the Leggett–Williams' fixed point theorem.

In [15], Sun considered the following second-order three-point boundary value problem

$$\begin{cases} x''(t) + a(t)f(t, x(t)) = 0, & 0 < t < 1, \\ x(0) = x(1) = \alpha x(\eta), \end{cases}$$

where $\alpha \in (0,1),\ \eta \in (0, \frac{1}{2}],\ a \in L^1(0,1)$ is nonnegative and symmetric on $(0,1),\ f : [0,1] \times [0,+\infty) \to [0,+\infty)$ satisfies Carathéodory conditions and $f(\cdot, x)$ is symmetric on [0,1] for all $x \in [0,+\infty)$. By using fixed point index theorems, the author get some optimal existence criteria for the existence of one or two symmetric solutions which involve the principal eigenvalue of a related linear operator.

At the same time, a class of boundary value problems with integral boundary conditions appeared in heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. Such problems include two, three, multi-point and nonlocal boundary value problems as special cases and attracted the attention of Gallardo [9], Karakostas and Tsamatos [10], Lomtatidze and Malaguti [11] and the references therein. For more information about the general theory of integral equations and their relation with boundary-value problems we refer to the book of Corduneanu [12] and Agarwal and O'Regan [13]. For some other excellent results and applications of the case that ordinary differential equation with integral boundary conditions to a variety of problems from Ahmad, Alsaedi and Alghamdi [35], Feng, Do and Ge [36], Feng, Ji and Ge [37], Infante and Webb [38], Kang, Wei and Xu [39], Ma [40], Webb [41], Webb and Infante [42,43], Yang [44], Zhang, Feng and Ge [45–47].

Motivated by the works mentioned above, we intend in this paper to study the existence and nonexistence of symmetric positive solutions of the following fourth-order boundary value problem with integral boundary conditions

$$\begin{cases} (u(t)x'''(t))' = w(t)f(t, x(t)), & 0 < t < 1, \\ x(0) = x(1) = \int_0^1 g(s)x(s)ds, \\ ax''(0) - b\lim\limits_{t\to 0^+} u(t)x'''(0) = \int_0^1 h(s)x''(s)ds, \\ ax''(1) + b\lim\limits_{t\to 1^-} u(t)x'''(1) = \int_0^1 h(s)x''(s)ds, \end{cases} \tag{1.1}$$

where $a, b > 0,\ u \in C^1([0,1] \to [0,+\infty))$ is symmetric on $[0,1], w \in L^p[0,1]$ for some $1 \leqslant p \leqslant +\infty$, and is symmetric on the interval [0,1], $f : [0,1] \times [0,+\infty) \to [0,+\infty))$ is continuous, and $f(1-t, x) = f(t, x)$ for all $(t, x) \in [0,1] \times [0,+\infty)$, and $g, h \in L^1[0,1]$ are nonnegative, symmetric on [0,1].

For the case of $a = 1,\ b = 0,\ p(t) \equiv 1$, and $w : [0,1] \to [0,+\infty)$ is continuous, symmetric on the interval [0,1], problem (1.1) reduces to the problem studied by Ma in [40]. By using the fixed point index in cones, the author obtained the existence of at least one symmetric positive solution.

Our approach is similar to that used in [4,5,40,45], i.e., Krasnoselskiis fixed point theorem is employed as the main tool of analysis, but the details here are not trivial. For example, in order to apply the ideas in [4,5,40,45] to our problem, we need to obtain some special lower and upper bounds for the Green's function of the problem (1.1) (see Proposition 2.11 and Proposition 2.12 in Section 2 below). To establish such bounds, very technical arguments are involved due to the fact that we have a general integral boundary conditions. By comparison, our results seem more natural than those in [40,45], and in this case, the results in [40,45] are special cases of those in this paper. Our work extends and complements many results such as those in [4,5,40,45,48–60,64,65]. On the other hand, as far as fourth order boundary value problems are concerned, a great deal of existence and uniqueness results have been established up to now. For details, see, for example, [3–5,19,21,22,30,48–60,64,65] and the references therein. However, among the existing results no one can be applied to our problem. This is another reason why we study problem (1.1). Finally, when it comes to the existence of symmetric positive solutions for fourth order boundary value problems, all the existence results obtained in previous papers are for the case $w \in C[0,1]$ or $w \in C(0,1)$, not for the case $w \in L^p[0,1]$ for some $1 \leqslant p \leqslant +\infty$, so it is interesting and important to study the existence of symmetric positive solutions for problem (1.1).