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Spectral parameter power series for fourth-order Sturm–Liouville problems

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ABSTRACT

A general solution of the fourth-order Sturm–Liouville equation is presented in the form of a spectral parameter power series (SPPS). The uniform convergence of the series is proved and the coefficients of the series are calculated explicitly through a recursive intergration procedure. Based on the SPPS representation characteristic equations for spectral problems arising in mechanics and elasticity theory are obtained and it is shown that the spectral problems reduce to computation of zeros of corresponding analytic functions of the spectral parameter given by their Taylor series expansions. This leads to a simple and efficient numerical method for solving the spectral problems for fourth-order Sturm–Liouville equations. Several examples of application are discussed.

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1. Introduction

A representation based on spectral parameter power series (SPPS) for solutions of second order linear differential equations was introduced in [19] as an application of pseudoanalytic (generalized analytic) function theory [20] and applied to a variety of related spectral problems [8,9,12,17,18,21,23] and others. The solution of the Sturm–Liouville type equation is represented as a power series with respect to the spectral parameter, and the expansion coefficients are obtained by a simple recursive procedure. The main advantage of the SPPS representations in application to spectral problems consists in the possibility of writing down in the explicit form the characteristic (dispersion) equation of the problem. Under certain (usually, well known conditions) it has the form $\Phi(\lambda) = 0$ where Φ is an analytic function of the complex variable λ. In many problems the existence of such complex analytic function Φ is a well established fact, and precisely the SPPS approach allows one to obtain the characteristic function Φ in the form of a power series with easily calculated expansion coefficients independent of λ. The numerical results (see the review [17]) show that the SPPS method is a powerful, easy in use and highly competitive tool for practical solution of spectral problems of mathematical physics.

In the present work we do not only apply the SPPS method to fourth-order Sturm–Liouville problems with very satisfactory results but also develop several new ideas which have not been introduced previously even for second-order problems. Namely, we establish that the SPPS method is applicable to equations in which the spectral parameter is a multiplier of a linear differential operator of a lower order than the main term of the equation, and show that it is convenient to use the SPPS method in a combination with the principle of the argument, a classical result from complex analysis. Both developments are especially important in the case of the fourth-order Sturm–Liouville problems. The equations with λ multiplied by a linear differential operator frequently lead to the appearance of complex eigenvalues, and this case cannot be efficiently solved by standard numerical techniques using in different ways the shooting method. Thus, the SPPS approach offers a

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unique possibility to solve such type of problems accurately and in a quite simple way. The principle of the argument combined with SPPS allows one to compute a number of eigenvalues in an arbitrary domain of interest in complex plane of the variable λ. This a priori knowledge of the number of the eigenvalues is obviously helpful for calculating them with the aid of the SPPS method.

We consider the fourth-order linear differential equations of the form

$$(pu'')'' + (qu')' = \lambda ru \tag{1}$$

and more generally,

$$(pu'')'' + (qu')' = \lambda R[u], \tag{2}$$

where p, q, r and u are assumed to be complex-valued functions of the real variable x, R is a linear differential operator of the order $n \leqslant 3$ and $\lambda \in \mathbb{C}$ is a spectral parameter. The conditions on the regularity of the coefficients in the considered equations are specified below (Theorem 3 and Remark 3). We are primarily interested in Sturm–Liouville type spectral problems for Eqs. (1) and (2) on a finite interval. In general we do not assume the self-adjointness of the corresponding problems and consider not only complex-valued coefficients but also boundary conditions involving complex parameters.

We obtain a representation of the general solution of (1) and (2) in the form of SPPS and apply it to get characteristic equations explicitly as well as to approximate the eigenvalues and eigenfunctions of corresponding Sturm–Liouville problems. Numerical solution of fourth-order Sturm–Liouville equations and spectral problems is a subject of a considerable number of recent publications (see, e.g. [1,2,4,7,10,14,25,27–32]). As we show in the present work the SPPS method is highly competitive in comparison with other techniques and moreover, it is applicable in the situations in which other methods are not, especially when there are complex eigenvalues.

The left-hand side of (1) and (2) is a differential expression of a special form, frequently studied in physically meaningful applications (see, e.g. [3,6,11]). The restriction of the present work to equations of the form (1) and (2) is due to a special form of a factorization of the operator which is behind of the SPPS representation (see Proposition 1). We expect to be able to extend the SPPS approach onto more general linear fourth-order differential equations in some future work.

In Section 2 we present the SPPS form of the general solution of (1) and (2). In Section 3 we show its application to spectral problems obtaining corresponding characteristic functions and approximating the eigenvalues. The SPPS method enables one with the possibility to visualize the characteristic function of the spectral problem which sometimes may be of more interest than even its solution. We illustrate this possibility on some of the numerical examples presented.

2. Solution of the Sturm–Liouville equation

In the following auxiliary statement we introduce a convenient factorized form of the linear differential operator on the left-hand side of (1) and (2).

Proposition 1. *Let the equation*

$$(pv')' + qv = 0, \tag{3}$$

with $p \in C^2[a,b]$ admit a particular (complex-valued) solution f such that $f \in C^3(a,b) \cap C[a,b]$, $1/f \in C[a,b]$. Then the operator on the left-hand side of Eqs. (1) and (2), $L := \frac{d^2}{dx^2} p \frac{d^2}{dx^2} + \frac{d}{dx} q \frac{d}{dx}$ can be factorized as follows

$$L = \frac{d}{dx}\left(\frac{1}{f}\frac{d}{dx}\left(pf^2\frac{d}{dx}\left(\frac{1}{f}\frac{d}{dx}\right)\right)\right). \tag{4}$$

Proof. Direct application of the operator L to an admissible function u leads to the following equalities

$$pf^2\frac{d}{dx}\left(\frac{1}{f}\frac{du}{dx}\right) = p(fu'' - f'u')$$

and

$$\frac{1}{f}\frac{d}{dx}\left(pf^2\frac{d}{dx}\left(\frac{1}{f}\frac{du}{dx}\right)\right) = \frac{1}{f}\frac{d}{dx}(p(fu'' - f'u')) = p'u'' + qu' + pu'''.$$

Thus,

$$\frac{d}{dx}\left(\frac{1}{f}\frac{d}{dx}\left(pf^2\frac{d}{dx}\left(\frac{1}{f}\frac{du}{dx}\right)\right)\right) = p''u'' + p'u''' + q'u' + qu'' + p'u''' + pu^{(4)} = \frac{d}{dx}(p'u'' + pu''') + \frac{d}{dx}(qu') = (pu'')'' + (qu')'. \quad \square$$

Remark 2. Factorizations of such kind sometimes are called Polya factorizations [16]. Here an important detail is that all the involved functions including the particular solution f are allowed to take complex values and to the difference of more

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