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Oscillation theorems for higher order neutral differential equations ☆

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ABSTRACT

The aim of this paper is to study the asymptotic properties and oscillation of the nth order neutral differential equations

$$\left(r(t)[x(t)+p(t)x(\tau(t))]^{(n-1)}\right)'+q(t)x(\sigma(t))=0. \tag{E}$$

Obtained results are based on the new comparison theorems, that permit to reduce the problem of the oscillation of the nth order equation to the oscillation of a set of the first order equation. Obtained comparison principles essentially simplify the examination of studied equations and allow to relax some conditions imposed on the coefficients of (E).

1. Introduction

In this paper, we shall study the asymptotic and oscillation behavior of the solutions of the nth order ($n \geqslant 3$) neutral differential equations of the form

$$\left(r(t)[x(t)+p(t)x(\tau(t))]^{(n-1)}\right)'+q(t)x(\sigma(t))=0, \tag{E}$$

where $q(t) \in C([t_0,\infty)), r(t), p(t), \tau(t), \sigma(t) \in C^1([t_0,\infty))$ and

(H_1) $r(t)>0,\ q(t)>0,\ 0 \leqslant p(t) \leqslant p_0 < \infty$;
(H_2) $\lim_{t\to\infty}\tau(t)=\infty,\ \lim_{t\to\infty}\sigma(t)=\infty,\ \sigma(t)<t, \sigma(t)$ nondecreasing;
(H_3) $\tau'(t) \geqslant \tau_0 > 0, \tau\circ\sigma=\sigma\circ\tau$.

For our further references, we denote and assume that

$$R(t)=\int_{t_0}^{t}\frac{1}{r(s)}\mathrm{d}s \to \infty \text{ as } t\to\infty. \tag{1.1}$$

We set $z(t)=x(t)+p(t)x(\tau(t))$. By a solution of Eq. (E), we mean a function $x(t)\in C([t_0,\infty))$, such that $z(t)\in C^{n-1}([t_0,\infty))$, and $r(t)z^{(n-1)}(t)\in C^1([t_0,\infty))$ and $x(t)$ satisfies (E) on $[t_0,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geqslant T\}>0$ for all $T \geqslant t_0$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$ and otherwise, it is said to be nonoscillatory. Eq. (E) itself is said to be oscillatory if all its solutions are oscillatory.

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Recently, great effort has been devoted to examination of the neutral differential equations, see [1–16]. The authors have studied mainly properties of the second order equations. Grammatikopoulos et al. [9] showed that $0 \leqslant p(t) \leqslant 1$ together with $\int^{\infty} q(s)(1 - p(s - \sigma))\, ds = \infty$ guarantee the oscillation of the neutral equation

$$(x(t) + p(t)x(t - \tau))'' + q(t)x(t - \sigma) = 0.$$

For the same equation Erbe et al. [7] established the oscillation criterion that requires

$$q(t) \geqslant q_0 > 0, \quad p_1 \leqslant p(t) \leqslant p_2, \quad p(t) \quad \text{not eventually negative}.$$

This result has been improved and generalized by other authors. We mention Grace and Lalli [8] who studied the oscillation of

$$\left(r(t)[x(t) + p(t)x(t - \tau)]'\right)' + q(t)f(x(t - \sigma)) = 0,$$

under the conditions

$$\frac{f(x)}{x} \geqslant k, \quad \int^{\infty} \frac{ds}{r(s)} = \infty$$

and

$$\int^{\infty} \rho(s)q(s)(1 - p(s - \sigma)) - \frac{(\rho'(s))^2 r(s - \sigma)}{4k\rho(s)}\, ds = \infty,$$

where $\rho(t)$ is an optional function.

Zafer in [15] discussed oscillation criteria for the higher order equation

$$[x(t) + p(t)x(\tau(t))]^{(n)} + f(t, x(t), x(\sigma(t))) = 0,$$

with $0 \leqslant p(t) \leqslant 1$.

Zhang et al. [16] considered the oscillation of the even order nonlinear neutral equation

$$[x(t) + p(t)x(\tau(t))]^{(n)} + q(t)f(x(\sigma(t))) = 0,$$

where again $0 \leqslant p(t) \leqslant 1$.

The same type of differential equation has been discussed by Li et al. [13], who attempted to relax some restriction imposed on the coefficients in earlier papers.

In the paper, we shall study properties of the higher order neutral differential equations. We generalize some earlier results known for the second and the third order neutral differential equations and relax some conditions that are in generally imposed on the coefficients of neutral equations. See [1–16].

For the particular case when n is odd and the gap between t and $\sigma(t)$ is small, then there exists a nonoscillatory solution of (E) and so in this case our goal is to prove that every nonoscillatory solution of (E) tends to zero as $t \to \infty$. While if difference $t - \sigma(t)$ is large enough then we shall study the oscillation of (E). So our aim in this article is to study both the above mentioned cases and to provide a general classification of the nonoscillatory solutions of the studied equation.

Various techniques appeared for the investigation the neutral differential equations. Our method is based on the establishing new comparison theorems for comparing the nth order equation (E) with a set of the first order delay differential equations in the sense, that the oscillations of these equations yield the oscillation of (E). Established comparison theorems essentially simplify the examination of (E) and enable us also to eliminate some conditions imposed in the cited papers on the coefficients of (E). Moreover, our results can be easily extend to cover also the more general differential equations.

Remark 1. All functional inequalities considered in this paper are assumed to hold eventually, that is they are satisfied for all t large enough.

Remark 2. Without loss of generality, we can deal only with the positive solutions of (E).

2. Main results

We begin with the classification of possible positive solutions of (E). So assume that $x(t)$ is a positive solution of (E), then we say that the corresponding function $z(t) = x(t) + p(t)x(\tau(t))$ is of degree ℓ if

$$z^{(i)}(t) > 0, \quad \text{for} \quad 0 \leqslant i \leqslant \ell, \tag{2.1}$$

$$(-1)^{i-\ell} z^{(i)}(t) > 0, \quad \text{for} \quad \ell \leqslant i \leqslant n - 1, \tag{2.2}$$

$$\left(r(t)z^{(n-1)}(t)\right)' < 0, \tag{2.3}$$

eventually. For our incoming references, we denote the set of all corresponding functions $z(t)$ of degree ℓ by N_ℓ.