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The symmetric Sinc-Galerkin method yields ADI model problems

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ABSTRACT

We show that when a symmetric Sinc-Galerkin method is used to solve a Poisson problem, the resulting Sylvester matrix equation is a discrete ADI model problem. We employ a new alternating direction scheme known as the alternating-direction Sinc-Galerkin (ADSG) method on illustrative partial differential equation boundary-value problems to document the exponential convergence rate that can be achieved. Unlike classical ADI schemes, direct numerical application of ADSG avoids the computation of iteration parameters, matrix eigenvalues and eigenvectors, as well as the use of the Kronecker product and sum.

1. Introduction

Since their introduction in [13], sinc methods have been used to solve a variety of differential equations [2,7–10,14,15]. Their exponential convergence rate has made such methods excellent tools for accurately approximating the solution to partial differential equation boundary-value problems. The application of sinc methods to a Poisson problem leads to matrix systems in the form of Sylvester or Lyapunov equations.

Alternating-direction implicit (ADI) methods were originally discussed in [12,3]. The simple structure of the matrix systems resulting from their application greatly simplified the numerical solution of parabolic and elliptic problems.

In 1984 it was recognized that the Lyapunov matrix equation

$$AX + XA^T = C$$

is a *discrete* ADI *model problem* [5]. For such problems, ADI methods are very efficient. In this paper we extend this result and prove that application of the symmetric Sinc-Galerkin method to a partial differential equation boundary-value problem yields a Sylvester matrix equation

$$AX + XB = C$$

that is also a discrete ADI model problem.

In Section 2 we provide a brief description of the classical ADI scheme and describe what is meant by a discrete ADI model problem. In Section 3 we discuss the symmetric Sinc-Galerkin method and use it to solve a general Poisson problem on the unit square. Section 4 is devoted to a proof of the main result of this paper. In Section 5 we briefly describe the alternating-direction Sinc-Galerkin (ADSG) method which we apply to two specific elliptic problems in Section 6.

2. ADI iteration

As described in [6], alternating-direction iteration may be used to solve the linear system

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$$M\mathbf{y} = \mathbf{b} \tag{1}$$

when the matrix M is symmetric positive definite (SPD), M can be expressed as a sum of SPD matrices H and V, $M = H + V$, and numerical implementation of the following algorithm

$$\begin{aligned} &\mathbf{y}_0 = 0, \\ &(H + p_j I)\mathbf{y}_{j-\frac{1}{2}} = \mathbf{b} - (V - p_j I)\mathbf{y}_{j-1}, \\ &(V + q_j I)\mathbf{y}_j = \mathbf{b} - (H - q_j I)\mathbf{y}_{j-\frac{1}{2}}, \quad j = 1, 2, \ldots, J \end{aligned} \tag{2}$$

is efficient. When the eigenvalues of M are in the positive-real half-plane (which is certainly the case for an SPD matrix M) and, in addition, H and V commute, the system (1) is classified as a *discrete* ADI *model problem* and (2) is very efficient.

3. The symmetric Sinc-Galerkin method

The Galerkin method is a variational method for approximating the solution of differential equations based on the weak form of the differential equation. Sinc-Galerkin methods employ sinc basis functions to form those approximations. Symmetrization of the Sinc-Galerkin method was introduced in [7]. Application of the symmetric Sinc-Galerkin method to a general self-adjoint Poisson problem results in a discrete sinc system which may be expressed in matrix form as a Sylvester equation.

Let $\mathbb{C}$ denote the set of all complex numbers and for all $z \in \mathbb{C}$ define the *sine cardinal* or *Sinc* function by

$$\operatorname{sinc}(z) \equiv \begin{cases} \frac{\sin(\pi z)}{\pi z} & z \neq 0 \\ 1 & z = 0 \end{cases} \tag{3}$$

For $h > 0$ and any integer k, the translated sinc function with evenly spaced nodes is denoted $S(k,h)(z)$ and defined by

$$S(k,h)(z) = \operatorname{sinc}\left(\frac{z - kh}{h}\right) \tag{4}$$

Consider the Poisson problem

$$\begin{aligned} &-\triangle u(x,y) \equiv -(u_{xx} + u_{yy}) = f(x,y), \quad (x,y) \in \Omega = (0,1) \times (0,1) \\ &u(x,y) = 0, \quad (x,y) \in \partial\Omega \end{aligned} \tag{5}$$

and assume a unique solution exists. Let M_x, N_x, M_y and N_y be positive integers and assume the approximate solution to (5) takes the form

$$u^s_{m_x,m_y} = \sum_{j=-M_y}^{N_y} \sum_{i=-M_x}^{N_x} u^s_{ij} S_{ij}(x,y) \tag{6}$$

where

$$m_x = M_x + N_x + 1, \quad m_y = M_y + N_y + 1 \tag{7}$$

and the basis functions $\{S_{ij}(x,y)\}$, $-M_x \leqslant i \leqslant N_x$, $-M_y \leqslant j \leqslant N_y$ are given by

$$S_{ij}(x,y) = [S(i,h_x) \circ \phi_x(x)][S(j,h_y) \circ \phi_y(y)] \tag{8}$$

for

$$\phi_z(z) = \ln\left(\frac{z}{1-z}\right) \tag{9}$$

where $z = x$ or y. Note that $\phi_z(z)$ is a conformal map of the unit interval onto the real line. Define the inner product by

$$(f,g) = \int_0^1 \int_0^1 f(x,y) g(x,y) v(x) w(y) dx dy \tag{10}$$

where the product $v(x)w(y)$ plays the role of a weight function. The choice of weight function can have a significant impact on the difficulty of solving the matrix systems that result from the application of sinc methods. The choice

$$v(x)w(y) = \frac{1}{\phi'_x(x)\phi'_y(y)}$$

which is thoroughly discussed in [14], facilitates eliminating boundary terms in the quadrature rules used to approximate inner products. The choice

$$v(x)w(y) = \frac{1}{\sqrt{\phi'_x(x)\phi'_y(y)}} \tag{11}$$