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Boundary value problems for the diffusion equation of the variable order in differential and difference settings

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ABSTRACT

Solutions of boundary value problems for a diffusion equation of fractional and variable order in differential and difference settings are studied. It is shown that the method of the energy inequalities is applicable to obtaining a priori estimates for these problems exactly as in the classical case. The credibility of the obtained results is verified by performing numerical calculations for a test problem.

1. Introduction

Fractional calculus is used for the description of a large class of physical and chemical processes that occur in media with fractal geometry as well as in the mathematical modeling of economic and social-biological phenomena [1–5]. In general, a medium in which a process proceeds is not homogenous, moreover, its properties may vary in time. Mathematical models containing equations with variable order derivatives provide a more accurate and realistic description of processes proceeding in such complex media (see e.g. [6–8]).

Therefore, the development of numerical and analytical methods of the theory of fractional order differential equations is an actual and important problem.

Numerical methods for solving variable order fractional differential equations with various kinds of the variable order fractional derivative have been proposed [9–14].

The positivity of the fractional derivative operator has been proved in [1] and this result allows to obtain a priori estimates for solutions of a large class of boundary value problems for the equations containing fractional derivatives. The authors of the paper [15] have obtained a priori estimate for the solution of the Dirichlet boundary value problem of a fractional order diffusion equation in terms of a fractional Riemann–Liouville integral. The fractional diffusion equation with the regularized fractional derivative has been studied, for example, in [16]. In the papers [17–19], the diffusion-wave equation with Caputo and Riemann–Liouville fractional derivatives has been studied. The difference schemes for boundary value problems for the fractional diffusion equation both in one and multidimensional cases have been studied [15,20,21]. A priori estimates for the difference problems obtained in [15,20,21] by using the maximum principle imply the stability and convergence of the considered difference schemes.

Using the energy inequality method, a priori estimates for the solution of the Dirichlet and Robin boundary value problems for the diffusion-wave equation with Caputo fractional derivative have been obtained [22]. More references on fractional order differential equations, including the diffusion-wave equation, can be found, for example, in [23].

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2. Boundary value problems in differential setting

2.1. The Dirichlet boundary value problem

In rectangle $\overline{Q}_T = \{(x,t) : 0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T\}$ let us study the boundary value problem

$$\partial_{0t}^{\alpha(x)} u = \frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right) - q(x,t)u + f(x,t), \quad 0 < x < l,\ 0 < t \leqslant T, \tag{1}$$

$$u(0,t) = 0, \quad u(l,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{2}$$

$$u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant l, \tag{3}$$

where $0 < c_1 \leqslant k(x,t) \leqslant c_2,\ q(x,t) \geqslant 0,\ \partial_{0t}^{\alpha(x)} u(x,t) = \int_0^t u_\tau(x,\tau)(t-\tau)^{-\alpha(x)} d\tau / \Gamma(1-\alpha(x))$ is a Caputo fractional derivative of order $\alpha(x)$, $0 < \alpha(x) < 1$, for all $x \in (0,l), \alpha(x) \in C(0,T)$ [18,24].

Suppose further the existence of a solution $u(x,t) \in C^{2,1}(\overline{Q}_T)$ for the problem (1)–(3), where $C^{m,n}$ is the class of functions, continuous together with their partial derivatives of the order m with respect to x and order n with respect to t on $\overline{Q}_T$.

The existence of the solution for the initial boundary value problem of a number of fractional order differential equation has been proved [25–27].

Let us prove the following:

Lemma 1. *For any functions $v(t)$ and $w(t)$ absolutely continuous on $[0,T]$, one has the equality:*

$$v(t)\partial_{0t}^{\beta} w(t) + w(t)\partial_{0t}^{\beta} v(t) = \partial_{0t}^{\beta}(v(t)w(t)) + \frac{\beta}{\Gamma(1-\beta)}\int_0^t \frac{d\xi}{(t-\xi)^{1-\beta}}\int_0^{\xi}\frac{v'(\eta)d\eta}{(t-\eta)^{\beta}}\int_0^{\xi}\frac{w'(s)ds}{(t-s)^{\beta}}, \tag{4}$$

where $0 < \beta < 1$.

Proof. Let us consider the difference

$$\begin{aligned}
v(t)\partial_{0t}^{\beta} w(t) + w(t)\partial_{0t}^{\beta} v(t) - \partial_{0t}^{\beta}(v(t)w(t)) &= \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{w'(s)(v(t)-v(s)) + v'(s)(w(t)-w(s))}{(t-s)^{\beta}} ds \\
&= \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{v'(s)ds}{(t-s)^{\beta}}\int_s^t w'(\xi)d\xi + \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{w'(s)ds}{(t-s)^{\beta}}\int_s^t v'(\xi)d\xi \\
&= \frac{1}{\Gamma(1-\beta)}\int_0^t w'(\xi)d\xi\int_0^{\xi}\frac{v'(s)ds}{(t-s)^{\beta}} + \frac{1}{\Gamma(1-\beta)}\int_0^t v'(\xi)d\xi\int_0^{\xi}\frac{w'(s)ds}{(t-s)^{\beta}} \\
&= \frac{1}{\Gamma(1-\beta)}\int_0^t (t-\xi)^{\beta}\frac{\partial}{\partial \xi}\left(\int_0^{\xi}\frac{v'(\eta)d\eta}{(t-\eta)^{\beta}}\int_0^{\xi}\frac{w'(s)ds}{(t-s)^{\beta}}\right)d\xi \\
&= \frac{\beta}{\Gamma(1-\beta)}\int_0^t \frac{d\xi}{(t-\xi)^{1-\beta}}\int_0^{\xi}\frac{v'(\eta)d\eta}{(t-\eta)^{\beta}}\int_0^{\xi}\frac{w'(s)ds}{(t-s)^{\beta}}.
\end{aligned}$$

The proof of the lemma is complete. □

If $v(t) = w(t)$ then from the Lemma 1 one has the following:

Corollary 1. *For any function $v(t)$ absolutely continuous on $[0,T]$, the following equality takes place:*

$$v(t)\partial_{0t}^{\beta} v(t) = \frac{1}{2}\partial_{0t}^{\beta} v^2(t) + \frac{\beta}{2\Gamma(1-\beta)}\int_0^t \frac{d\xi}{(t-\xi)^{1-\beta}}\left(\int_0^{\xi}\frac{v'(\eta)d\eta}{(t-\eta)^{\beta}}\right)^2, \tag{5}$$

where $0 < \beta < 1$.

Let us use the following notation: $\|u\|_0^2 = \int_0^l u^2(x,t)dx$, $D_{0t}^{-\beta}u(x,t) = \int_0^t (t-s)^{\beta-1}u(x,s)ds/\Gamma(\beta)$ – fractional Riemann–Liouville integral of order β.

Theorem 1. *If $k(x,t) \in C^{1,0}(\overline{Q}_T), q(x,t), f(x,t) \in C(\overline{Q}_T)$, $k(x,t) \geqslant c_1 > 0, q(x,t) \geqslant 0$ everywhere on $\overline{Q}_T$, then the solution $u(x,t)$ of the problem (1)–(3) satisfies the a priori estimate:*

$$\int_0^l D_{0t}^{\alpha(x)-1}u^2(x,t)dx + c_1\int_0^t \|u_x(x,s)\|_0^2 ds \leqslant \frac{l^2}{2c_1}\int_0^t \|f(x,s)\|_0^2 ds + \int_0^l \frac{t^{1-\alpha(x)}}{\Gamma(2-\alpha(x))}u_0^2(x)dx. \tag{6}$$