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Solutions converging to zero of some systems of nonlinear functional differential equations with iterated deviating argument

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ABSTRACT

Some sufficient conditions for the existence of a p-parameter family of solutions converging to zero of a system of nonlinear functional differential equations with an iterated deviating argument dependent of unknown function, are given.

1. Introduction

Nonlinear differential or functional differential equations, not solved (or partially solved) with respect to the highest-order derivatives, have been studied a lot, see, for example, [1–16,29] and the references therein.

The next system of functional differential equations

$$x'(t) = Ax(t) + F(t, x(t), x(f(t, x(t))), x'(g(t))), \tag{1}$$

where $F(t, x, y, z)$, $f(t, x)$, and $g(t)$ are real-valued functions continuous for $t \in \mathbb{R}$ and $x, y, z \in \mathbb{R}^N$, is one which has attracted some attention among the experts in the research area.

Iteration methods for approximating fixed points which are iterations of some iterative processes were introduced in our papers [17–21]. Similar ideas in the theory of difference equations can be found, for example, in [22–25]. This idea motivated us to propose studying equations with continuous arguments, whose deviations of an argument depend on an unknown function which depend also of the function and so on (see [26–28]).

Here we continue this line of research by studying the following system of nonlinear functional differential equations

$$x'(t) = Ax(t) + F(t, x(t), x(v_1(t, x)), x'(g(t))), \tag{2}$$

where

$$v_j(t, x) = \varphi_j(t, x(\varphi_{j+1}(t, \dots x(\varphi_k(t, x(t))) \dots))), \quad j = \overline{1, k}, \tag{3}$$

$A = \operatorname{diag}(A_1, A_2)$, A_1 and A_2 are real constant matrices of dimensions $p \times p$ and $p' \times p'$, respectively, $p + p' = N, F : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $\varphi_j : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $j = \overline{1, k}$, $g : \mathbb{R} \to \mathbb{R}$, and $x(t)$ is an unknown function. Related results can be found, for example, in [2,6,9,11,13–15,26].

Our aim is to study solutions of system (2) converging to zero as $t \to +\infty$, under some assumptions which guarantee that $x(t) \equiv 0$ is a solution of the system.

Throughout this paper $|\cdot|$ denote the modulus of a number, or a norm of a vector in $\mathbb{R}^N$, or a norm of a matrix, depending of which kind of quantity appears under the symbol. Recall that all the norms in a finite dimensional space are equivalent so we can choose any of them, for example, the Euclidean one.

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2. Main result

In this section we formulate and prove the main result in this paper. The main result gives some sufficient conditions for the existence of a p-parameter family of continuously differentiable solutions of system of nonlinear functional differential equations (2) converging to zero as $t \to +\infty$.

Theorem 1. *Assume that the following conditions are satisfied:*

(a) *the inequalities*

$$|e^{A_1 t}| \leqslant N_1 e^{-\alpha_1 t} \quad \text{and} \quad |e^{-A_2 t}| \leqslant N_2 e^{-\alpha_2 t}, \tag{4}$$

hold for every $t \in \mathbb{R}_+ = [0, \infty)$ and for some positive constants $\alpha_1,\ \alpha_2,\ N_1$, and N_2;

(b) *functions $F(t,x,y,z),\ \varphi_j(t,x), g(t),\ j = \overline{1,k}$ are continuous for $t \in \mathbb{R},\ x,y,z \in \mathbb{R}^N$, and*

$$F(t,0,0,0) = 0, \quad t \in \mathbb{R}_+; \tag{5}$$

(c) *functions $F(t,x,y,z)$, $\varphi_j(t,x),\ j = \overline{1,k}$, satisfy the following Lipschitz conditions*

$$|F(t,x_1,y_1,z_1) - F(t,x_2,y_2,z_2)| \leqslant L_1|x_1 - x_2| + L_2|y_1 - y_2| + L_3|z_1 - z_2|, \tag{6}$$

$$|\varphi_j(t,x_1) - \varphi_j(t,x_2)| \leqslant l_j|x_1 - x_2|, \quad j = \overline{1,k}, \tag{7}$$

where $L_i,\ l_j,\ i \in \{1,2,3\},\ j = \overline{1,k}$, are positive constants, $(t,x_1,y_1,z_1),\ (t,x_2,y_2,z_2) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, and $(t,x_1),(t,x_2) \in \mathbb{R} \times \mathbb{R}^N$;

(d) *there is a $q \in (0,1)$ such that the following inequalities hold*

$$\left(L_1 + L_3 + L_2 \sum_{j=0}^{k} \left(\frac{N_1}{1-q}\right)^j \prod_{i=1}^{j} l_i\right) \left(\frac{N_1}{\alpha_1 - \beta} + \frac{N_2}{\alpha_2 + \beta}\right) \leqslant q, \tag{8}$$

$$|A|q + L_1 + L_3 + L_2 \sum_{j=0}^{k} \left(\frac{N_1}{1-q}\right)^j \prod_{i=1}^{j} l_i \leqslant q \tag{9}$$

for sufficiently small $\beta \in (0, \alpha_1)$;

(e) *the following inequalities*

$$v_j(t,x) \geqslant t, j = \overline{1,k}, \quad \text{and} \quad g(t) \geqslant t \tag{10}$$

hold for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^N$.

Then, for sufficiently small $L_i,\ l_j,\ i \in \{1,2,3\},\ j = \overline{1,k}$, system (2) has a p-parameter family of solutions $x(t)$ continuously differentiable for $t \in \mathbb{R}_+$ and such that

$$\lim_{t \to \infty} |x(t)| = 0. \tag{11}$$

Proof. Consider the following system of integral equations

$$x(t) = X(t) + \int_0^{+\infty} G(t-\tau) F(\tau, x(\tau), x(\varphi_1(\tau, x(\varphi_2(\tau, \ldots x(\varphi_k(\tau, x(\tau))) \ldots))))), x'(g(\tau))) d\tau, \tag{12}$$

where $X(t) = diag(e^{A_1 t}c, 0),\ c \in \mathbb{R}^p$ (we may assume that $|c| \leqslant 1$) and

$$G(t) = \begin{cases} -\mathrm{diag}(0, e^{A_2 t}), & t < 0, \\ \mathrm{diag}(e^{A_1 t}, 0), & t > 0. \end{cases}$$

It is easy to see that

$$G(+0) - G(-0) = E,$$

where E denote the $N \times N$ identity matrix, and that

$$G'(t) = AG(t)$$

for every $t \neq 0$, from which it easily follows that every solution of system (12) is a solution of system (2).

To prove the theorem we use the method of successive approximations and define a sequence $(x_n(t))_{n \in \mathbb{N}_0}$ by

$$x_0(t) = 0, \quad x_0'(t) = 0, \tag{13}$$