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Convexity preserving interpolatory subdivision with conic precision

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ABSTRACT

The paper is concerned with the problem of shape preserving interpolatory subdivision. For arbitrarily spaced, planar input data an efficient non-linear subdivision algorithm reproducing conic sections and respecting the convexity properties of the initial data, is here presented. Significant numerical examples are included to illustrate the effectiveness of the proposed method and the smoothness of the limit curves.

1. Introduction and state of the art

Subdivision schemes constitute a powerful alternative for the design of curves and surfaces over the widely studied parametric and implicit forms. In fact, they offer a really versatile tool that is, at the same time, very intuitive and easy to use and implement. This is due to the fact that subdivision schemes are defined via iterative algorithms which exploit simple refinement rules to generate denser and denser point sequences that, under appropriate hypotheses, converge to a continuous, and potentially smooth, function.

In the univariate case, the iteration starts with a sequence of points denoted by $\mathbf{p}^0 = (\mathbf{p}_i^0 : i \in \mathbb{Z})$, attached to the integer grid, and then for any $k \geqslant 0$ one subsequently computes a sequence $\mathbf{p}^{k+1} = S\mathbf{p}^k$, where $S : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ identifies the so-called subdivision operator.

Subdivision operators can be broadly classified into two main categories: *interpolating* and *approximating* [1,2]. Interpolating schemes are required to generate limit curves passing through all the vertices of the given polyline $\mathbf{p}^0$ (note that throughout the paper the terms "polygon" and "polyline" are used as an alternative to refer to point sequences). Therefore they are featured by refinement rules maintaining the points generated at each step of the recursion in all the successive iterations. Approximating schemes, instead, are not required to match the original position of vertices on the assigned polyline $\mathbf{p}^0$ and thus they adjust their positions aiming at very smooth and visually pleasing limit shapes that approximate it. As a consequence, while in the case of *approximating* subdivision the newly generated vertices are not on the limit shape, in the case of *interpolatory* subdivision, in every iteration a finer data set $\mathbf{p}^{k+1}$ is obtained by taking the old data values $\mathbf{p}^k$ and inserting new points in between them, so that the limit curve not only interpolates the initial set of points but also all the points generated through the whole process. Every such new point is calculated using a finite number of existing, usually neighboring points. In particular, if the computation of the new points is carried out through a linear subdivision operator S, the scheme is said to be *linear*, otherwise *non-linear*. Then, inside the above identified categories, the schemes can also be further classified. More specifically, the refinement rules of a scheme can be distinguished between *stationary* (when they do not alter from level to level) and *non-stationary*; between *uniform* (when they do not vary from point to point) and *non-uniform*; between *binary* (when they double the number of points at each iteration) and *N-ary*, namely of arity $N > 2$.

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Most of the univariate subdivision schemes studied in the literature are binary, uniform, stationary and linear. These characteristics, in fact, make it easier to study the mathematical properties of the limit curve, but seriously limit the applications of the scheme. Exceptions from a binary, or a uniform, or a stationary, or a linear approach, have already appeared (see for example [3] and references therein), but none of the methods proposed so far provides an interpolatory algorithm that can fulfill the list of *all* fundamental features considered essential in applications. These features can briefly be summarized as:

(i) generating a visually-pleasing limit curve which faithfully mimics the behavior of the underlying polyline without creating unwanted oscillations;
(ii) preserving the shape, i.e., the convexity properties of the given data;
(iii) identifying geometric primitives like circles and more generally conic sections, the starting polyline had been sampled from, and reproducing them.

Requirement (i) derives from the fact that, despite interpolating schemes being considered very well-suited for handling practical models to meet industrial needs (due to their evident link with the initial configuration of points representing the object to be designed), compared to their approximating counterparts, they are more difficult to control and tend to produce bulges and unwanted folds when the initial data are not uniformly spaced. Recently this problem has been addressed by using non-uniform refinement rules [4,5] opportunely designed to take into account the irregular distribution of the data. But, despite their established merit of providing visually pleasing results, there is no guarantee that such methods are convexity-preserving, i.e., that if a convex data set is given, a convex interpolating curve can be obtained. This is due to the fact that, such non-uniform schemes are linear and, as it is well-known [6], linear refinement operators that are C^1 cannot preserve convexity in general.

The property (ii) of convexity preservation is of great practical importance in modeling curves and surfaces tailored to industrial design (e.g. related to car, aeroplane or ship modeling where convexity is imposed by technical and physical conditions as well as by aesthetic requirements). In fact, if shape information such as convexity is not enforced, interpolatory curves, though smooth, may not be satisfactory as they may contain redundant wiggles and bumps rather than those suggested by the data points, i.e., they feature unacceptable visual artifacts. Preserving convexity, while a curve is interpolating a given data set, is far from trivial. But much progress has been made in this field, evidence of which is given by the recent burgeoning literature. In most publications, the introduction of subdivision schemes fulfilling requirement (ii) has been achieved through the definition of non-linear refinement rules. In fact, although linear subdivision schemes turn out to be simple to implement, easy to analyze and affine invariant, they have many difficulties to control the shape of the limit curve and avoid artifacts and undesired inflexions that usually occur when the starting polygon $\mathbf{p}^0$ is made of highly non-uniform edges. Non-linear schemes, instead, offer effective algorithms to be used in shape-preserving data interpolation [7–9,6,10–12].

On the basis of the well–known, linear Dubuc–Deslauriers interpolatory 4-point scheme [13], for example, several non-linear analogues have been presented in order to accomplish at least one of the three above listed properties. On the one hand, non-linear modifications of the classical 4-point scheme have been introduced to reduce the oscillations that usually occur in the limit curve when applying the refinement algorithm to polylines with short and long adjacent edges. These have been presented in [14,15], and as concerns the case of convexity-preserving strategies (which are the ones capable of completely eliminating the artifacts arising during the subdivision process), we find the papers [16,11]. On the other hand, for the purpose of enriching the Dubuc-Deslauriers 4-point scheme with the property (iii) of geometric primitives preservation, a non-linear 4-point scheme reproducing circles and reducing curvature variation for data off the circle, has been defined [17]. With the same intent, another modification of the classical 4-point scheme in a non-linear fashion, had been given in [18].

With these papers, the theoretical investigation of non-linear interpolatory subdivision has only begun. A lot is still to be done, in particular as concerns the use of non-linear rules for reproducing salient curves other than circles, considered of fundamental importance in several applications. So far, it has been shown that non-linear updating formulas can be used in the definition of non-stationary subdivision schemes aimed at reproducing polynomials and some common transcendental functions. In particular, [19] respectively [20–25] present subdivision algorithms that turn out to be circle-preserving respectively able to exactly represent any conic section. While the first is able to guarantee reproduction starting from given samples with any arbitrary spacing, for the latter ones the property of conic precision is confined to the case of equally-spaced samples. Most recently a shape and circle preserving scheme for any arbitrary data points has been presented in [7].

Therefore, an outstanding issue that should be considered is the possibility of defining an interpolatory subdivision scheme that is at the same time shape-preserving and artifact free, as well as capable of generating conic sections starting from any arbitrarily-spaced samples coming from a conic. This is exactly the purpose of this paper. Based on an approximation order four strategy presented in [26] for estimating tangents to planar convex data sequences, we propose a convexity-preserving interpolatory subdivision scheme with conic precision. This turns out to be a new kind of non-linear and geometry-driven subdivision method for curve interpolation.

The remainder of the paper is organized as follows. In Section 2 we start by describing the refinement strategy, which relies on a classical cross-ratio property for conic sections and uses the tangent estimator from [26], for the case of globally convex data. In Section 3 we adapt the scheme to general, not necessarily convex data by segmenting the given polygon into