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Remarks on the Hyers–Ulam stability of some systems of functional equations

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ABSTRACT

In this paper we present a method that allows to study the Hyers–Ulam stability of some systems of functional equations connected with the Cauchy, Jensen and quadratic equations. In particular we generalize and extend some already known results.

1. Introduction

Let I be a nonempty set, $(G,+)$ and $(G_i,+_i)$ for $i\in I$ be groupoids, i.e., nonempty sets endowed with binary operations $+:G^2\to G$ and $+_i:G_i^2\to G_i$, and $P_I:=\prod_{i\in I}G_i$ be the cartesian product of all the sets G_i, i.e.,

$$P_I=\left\{x:I\to\bigcup_{i\in I}G_i|x(i)\in G_i\text{ for }i\in I\right\}.$$

For any $j\in I, u\in G_j$ and $y\in P_I, y_u^j$ denotes the element of P_I that satisfies $y_u^j(i)=y(i)$ if $i\neq j$ and $y_u^j(j)=u$. Clearly, in the simplest case $I=\{1,2\}$ and $P_I=G_1\times G_2$, for each $y=(y_1,y_2)\in P_I, u\in G_1$ and $v\in G_2$, we have $y_u^1=(u,y_2)$ and $y_v^2=(y_1,v)$.

We say that a function $f:P_I\to G$ is *I-Cauchy* (or *I-homomorphism*) provided, for any $j\in I$ and $y\in P_I$, the function $f_j^y:G_j\to G$, given by

$$f_j^y(u)=f(y_u^j),\quad u\in G_j, \tag{1}$$

fulfils the condition

$$f_j^y(u+_jv)=f_j^y(u)+f_j^y(v),\quad u,v\in G_j. \tag{2}$$

In the particular case where $I=\{1,2\}$ and $(G_1,+_1),(G_2,+_2),(G,+)$ are groups, an I-Cauchy function $f:P_I\to G$ is quite often called a *biadditive mapping*; then condition (2), for $j=1,2$, takes the forms

$$f(y_1+_1y_1',y_2)=f(y_1,y_2)+f(y_1',y_2),\quad y_1,y_1',y_2\in G_1,$$

$$f(y_1,y_2+_2y_2')=f(y_1,y_2)+f(y_1,y_2'),\quad y_1,y_2,y_2'\in G_1.$$

Analogously, if $n\in\mathbb{N}$ ($\mathbb{N}$ stands for the set of all positive integers), $I=\{1,\ldots,n\}$ and $(G_1,+_1),\ldots,(G_n,+_n),(G,+)$ are groups, then an I-Cauchy function $f:P_I\to G$ is often called an *n-additive mapping*. Some basic facts on such mappings can be found for instance in [37], where their application to the representation of polynomial functions is also provided.

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A similar terminology can be applied for the Jensen and quadratic functional equations (for more information on these two equations see, e.g. [1,37,48]). Namely, we say that a function $f : P_I \to G$ is *I-Jensen* provided, for any $j \in I$ and $y \in P_I$, the function $f_j^y : G_j \to G$, defined by (1), satisfies the equation

$$2f_j^y\left(\frac{u +_j v}{2}\right) = f_j^y(u) + f_j^y(v), \quad u, v \in G_j, \tag{3}$$

(of course under the assumption of the divisibility by 2 with uniqueness in G_j, by which we mean that for each $x \in G_j$ there is a unique $z \in G_j$ (denoted by $\frac{x}{2}$) with $x = 2z := z +_j z$). Analogously, under the additional assumption that every $(G_j, +_j)$ is a group, we say that a function $f : P_I \to G$ is *I-quadratic provided*, for any $j \in I$ and $y \in P_I$, the function $f_j^y : G_j \to G$ is a solution of the quadratic equation

$$f_j^y(u +_j v) + f_j^y(u -_j v) = 2f_j^y(u) + 2f_j^y(v), \quad u, v \in G_j. \tag{4}$$

If we want to avoid these additional assumptions on G_j, we can replace (3) and (4), respectively, by the equations

$$2f_j^y(u +_j v) = f_j^y(2u) + f_j^y(2v), \quad u, v \in G_j \tag{5}$$

and

$$f_j^y(u +_j 2v) + f_j^y(u) = 2f_j^y(u +_j v) + 2f_j^y(v), \quad u, v \in G_j, \tag{6}$$

which allow us to consider them also on groupoids. Eq. (3) has exactly the same solutions as (5), e.g., under the assumption that $(G_j, +_j)$ is a commutative semigroup divisible by 2 with uniqueness (replace u and v by $2u$ and $2v$, respectively); the same concerns Eqs. (4) and (6) under the assumption that $(G_j, +_j)$ is a group (replace u by $u + v$). In our first theorem these latter forms of the Jensen and quadratic equations are motivations to formulate more general results.

Now, given disjoint sets $I_1, I_2, I_3 \subset I$ with $I_1 \cup I_2 \cup I_3 = I$, we can combine those three notions and say that a function $f : P_I \to G$ is *I_1-Cauchy, I_2-Jensen and I_3-quadratic* (briefly, *multi-Cauchy–Jensen-quadratic*) provided for every $y \in P_I$, the function f_j^y satisfies Eq. (2) for each $j \in I_1$, Eq. (5) for each $j \in I_2$ and Eq. (6) for each $j \in I_3$. If $I_3 = \emptyset$, then we simply say that such function is *I_1-Cauchy and I_2-Jensen* (briefly, *multi-Cauchy–Jensen*). Analogously, if $I_1 = \emptyset$ and/or $I_2 = \emptyset$, then we omit the parts *I_1-Cauchy* and/or *I_2-Jensen*, respectively.

Let us mention here that the notion of multi-Jensen function was introduced in 2005 by Prager and Schwaiger (see [44], and also [45]) in a connection with generalized polynomials, whereas Cauchy–Jensen mapping was defined by Park and Bae [42].

In this paper we study stability of the system of equations defining the multi-Cauchy–Jensen-quadratic mappings. Our results are significant supplements and/or generalizations of some classical outcomes from [2,3,10,12,21,22,25,26, 28,31,46,47] and recent results from [5–8,13–18,29,30,36,38,42,43] (in particular those concerning stability of the Cauchy–Jensen mappings from [8,29,30,38,42]).

Speaking of the stability of functional equations we follow the question raised in 1940 by S. M. Ulam when is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation? The first answer (in the case of Cauchy's functional equation in Banach spaces) to Ulam's question was given by Hyers (see [26]). After his result a great number of papers on the subject were published (see for instance [4,9,19,20,32,33,39,48] and the references given there), generalizing Ulam's problem and Hyers's theorem in various directions and to other functional equations (as the words *differing slightly* and *be close* may have various meanings, different kinds of stability can be dealt with).

2. Preliminaries

In the proofs we use a method suggested by Forti [23] (see also [24]) and explicitly presented in [11] on some examples. The main tool is the following result from [11] (it can be also easily derived from [24]).

Proposition 1. *Let (Y, d) be a complete metric space, K be a nonempty set, $f : K \to Y, \Psi : Y \to Y, a : K \to K$, $h : K \to [0, \infty), \lambda \in [0, \infty)$,*

$$d((\Psi \circ f \circ a)(x), f(x)) \leqslant h(x), \quad x \in K,$$

$$d(\Psi(x), \Psi(y)) \leqslant \lambda d(x, y), \quad x, y \in Y \tag{7}$$

and

$$H(x) := \sum_{i=0}^{\infty} \lambda^i h\left(a^i(x)\right) < \infty, \quad x \in K. \tag{8}$$

Then, for every $x \in K$, the limit

$$F(x) := \lim_{n\to\infty} (\Psi^n \circ f \circ a^n)(x)$$

exists and the function $F : K \to Y$, defined in this way, satisfies the equation