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The Nehari manifold for indefinite semilinear elliptic systems involving critical exponent

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ABSTRACT

In this paper, we study the combined effect of concave and convex nonlinearities on the number of solutions for an indefinite semilinear elliptic system $(E_{\lambda,\mu})$ involving critical exponents and sign-changing weight functions. Using Nehari manifold, the system is proved to have at least two nontrivial nonnegative solutions when the pair of the parameters (λ,μ) belongs to a certain subset of $\mathbb{R}^2$.

1. Introduction

In this paper we examine the multiplicity results derived from the nontrivial nonnegative solutions of the indefinite semilinear elliptic system below:

$$\begin{cases} -\Delta u = f_\lambda(x)\,|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}h(x)\,|u|^{\alpha-2}u\,|\,v|^{\beta} & \text{in } \Omega, \\ -\Delta v = g_\mu(x)\,|v|^{q-2}v + \frac{\beta}{\alpha+\beta}h(x)\,|u|^{\alpha}\,|\,v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \qquad (E_{\lambda,\mu})$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $0\in\Omega$, $\alpha>1,\ \beta>1$ satisfying $\alpha+\beta=2^*=\frac{2N}{N-2}\,(N\geqslant 3),\ 1<q<2$, and the parameters $\lambda,\ \mu\geqslant 0$. We assume that $f_\lambda(x)=\lambda f_+(x)+f_-(x)$ and $g_\mu(x)=\mu g_+(x)+g_-(x)$ where the weight functions f, g and h satisfy the following conditions:

$(D1)$ $f,g\in C(\overline{\Omega})$ with $\|f_+\|_\infty=\|g_+\|_\infty=1,\ f_\pm=\max\{\pm f,0\}\not\equiv 0$ and $g_\pm=\max\{\pm g,0\}\not\equiv 0$;

$(D2)$ $h\in C(\overline{\Omega})$ with the sets

$$\{x\in\Omega\mid h(x)\geqslant 0\}\cap\{x\in\Omega\mid f(x)>0\}$$

and

$$\{x\in\Omega\mid h(x)\geqslant 0\}\cap\{x\in\Omega\mid g(x)>0\}$$

having positive measures;

$(D3)$ there exists a positive number ρ with $\rho>2$ when $N\geqslant 6,\ \rho>(N-2)/2$ when $3\leqslant N\leqslant 5$ such that

$$h(0)=1=\max_{x\in\overline{\Omega}}h(x)$$

and

$$h(0)-h(x)=O(|\,x|^{\rho})\quad \text{as } x\to z.$$

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Remark 1.1. Let $B^N(0,r) = \{x \in \mathbb{R}^N \mid\mid x \mid< r\}$. Accordingly, by condition $(D3)$, we may assume that there exist two positive constants D_0 and r_0 such that

$$h(x) > 0 \quad \text{for all } x \in B^N(x_0, r_0) \subset \Omega$$

and

$$h(0) - h(x) \leqslant D_0 \mid x|^\rho \quad \text{for all } x \in B^N(x_0, r_0).$$

We propose to study the system $(E_{\lambda,\mu})$ in the framework of the Sobolev space $H = H_0^1(\Omega) \times H_0^1(\Omega)$ using the standard norm

$$\|(u,v)\|_H = \left(\int_\Omega \mid \nabla u|^2 + \int_\Omega \mid \nabla v|^2\right)^{\frac{1}{2}}$$

and consider, as a weak solution of the system, a pair of functions $(u,v) \in H$ if

$$\int_\Omega \nabla u \nabla \varphi_1 + \int_\Omega \nabla v \nabla \varphi_2 - \int_\Omega f_\lambda \,|u|^{q-2} u \varphi_1 - \int_\Omega g_\mu \mid v|^{q-2} v \varphi_2 - \frac{\alpha}{\alpha+\beta}\int_\Omega h\,|u|^{\alpha-2} u \mid v|^\beta \varphi_1 - \frac{\beta}{\alpha+\beta}\int_\Omega h \mid u|^\alpha \mid v|^{\beta-2} v \varphi_2 = 0 \quad \forall (\varphi_1, \varphi_2) \in H.$$

Subsequently, for $(u,v) \in H$, the associated energy functional is defined by

$$J_{\lambda,\mu}(u,v) = \frac{1}{2}\|(u,v)\|_H^2 - \frac{1}{q}\left(\int_\Omega f_\lambda \mid u|^q + \int_\Omega g_\mu \mid v|^q\right) - \frac{1}{\alpha+\beta}\int_\Omega h \mid u|^\alpha \mid v|^\beta.$$

The corresponding scalar version of semilinear elliptic equations with concave–convex nonlinearities, namely,

$$\begin{cases} -\Delta u = \lambda f(x) \mid u|^{q-2} u + h(x) \mid u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \qquad (E_\lambda)$$

has been widely studied with a plethora of results. For the case when the weight functions are taken to be constant, namely, $f \equiv h \equiv 1$ with $2 < p \leqslant 2^*$ in (E_λ), Ambrosetti et al. [3] demonstrated that there exists $\lambda_0 > 0$ such that at least two positive solutions are admitted for $\lambda \in (0, \lambda_0)$, a positive solution for $\lambda = \lambda_0$ while no positive solution exists if $\lambda > \lambda_0$. The problem was taken up by various authors for more general cases; the readers are referred to Ambrosetti et al. [2], Chen and Wu [14], de Figueiredo et al. [17], EL Hamidi [18], Lubyshev [24], Radulescu and Repovs [25] and Wu [29,30] for detailed results. In particular, extending the problem to consider sign-changing weight functions, the authors in [14,29,30] showed multiplicity results with respect to the parameter λ via the extraction of Palais–Smale sequences in the Nehari manifold.

For the systems of semilinear elliptic equations with concave–convex nonlinearities, various studies concerning the solutions structures have also been presented ([1,4,6,8,13,16,19,23,26,28]). Among these, Adriouch and EL Hamidi [4] considered the following system:

$$\begin{cases} -\Delta u = \lambda u + \frac{\alpha}{\alpha+\beta} \mid u|^{\alpha-2} u \mid v|^\beta & \text{in } \Omega, \\ -\Delta v = \mu \mid v|^{q-2} v + \frac{\beta}{\alpha+\beta} \mid u|^\alpha \mid v|^{\beta-2} v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

which has been proved to permit at least two positive solutions when the pair of parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^2$. Similar results were obtained by Hsu [21] of system $(E_{\lambda,\mu})$ when constant weight functions $f = g = h = 1$ were assumed. Further studies involving sign-changing weight functions were taken up by Hsu [22] and Wu [32], where the multiplicity results were obtained for the subcritical case $2 < \alpha + \beta < 2^*$ in [32] while those for the critical case $\alpha + \beta = 2^*$ were obtained in [22] for $\rho \geqslant N,\ N \geqslant 3$ with the constraint on one of the weight functions being positive. These results will be improved upon in this paper when we investigate further the multiplicity of nontrivial nonnegative solutions by considering the system given by $(E_{\lambda,\mu})$ extending the approach that was previously developed in [29,30,32]. The results presented here include cases for $\rho > 2$ when $N \geqslant 6$ and $\rho > (N-2)/2$ when $3 \leqslant N \leqslant 5$ while permitting the sign-changing property of all weight functions.

Taking S to be the best Sobolev constant for the embedding of $H_0^1(\Omega)$ in $L^{\alpha+\beta}(\Omega)$, the main result is stated in the following theorem.

Theorem 1.1. *Suppose that the weight functions f, g, h satisfy the conditions (D1)–(D3). Then there exists an explicit number* $C(\alpha,\beta,q,S) > 0$ *for which if the parameters* λ, μ *satisfy*

$$0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < C(\alpha,\beta,q,S),$$

then problem $(E_{\lambda,\mu})$ *has at least two solutions* $\left(u_{\lambda,\mu}^+, v_{\lambda,\mu}^+\right)$ *and* $\left(u_{\lambda,\mu}^-, v_{\lambda,\mu}^-\right)$ *such that* $u_{\lambda,\mu}^\pm \geqslant 0,\ v_{\lambda,\mu}^\pm \geqslant 0$ *in* Ω *and* $u_{\lambda,\mu}^\pm \neq 0,\ v_{\lambda,\mu}^\pm \neq 0$.

Note that the existence and multiplicity of solutions concerning other combined effects of nonlinearities or weight function profiles have also been studied by other authors, we refer the readers to [5,12,14,23] and the references therein.