Applied Mathematics and Computation 218 (2012) 10848–10860

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Collocation methods for nonlinear convolution Volterra integral equations with multiple proportional delays

Kai Zhang *, Jie Li, Haiming Song

Department of Mathematics, Jilin University, Changchun, Jilin 130023, PR China

ARTICLE INFO

Keywords:
Collocation methods
Nonlinear convolution
Volterra integral equation
Multiple proportional delays

ABSTRACT

In this paper, we apply the collocation methods to a class of nonlinear convolution Volterra integral equations with multiple proportional delays (NCVIEMPDs). We shall present the existence, uniqueness and regularity properties of analytic solution for this type equation, and then analyze the convergence and superconvergence properties of the collocation solution. The numerical results verify our theoretical analysis.

1. Introduction

Modeling many problems of physics, economics, stochastics, and other disciplines leads to nonlinear convolution Volterra integral equations with multiple proportional delays, where the rate of change of the process is not only determined by its present state but also by a certain past state. These are usually difficult to solve analytically and in many cases the solutions must be approximated. Numerical methods based on finite difference methods, Runge–Kutta methods, discontinuous Galerkin methods and spectral methods etc., have also been developed for various nonlinear Volterra integral equations and we refer to [1,3,5,7–9,11,17], and references therein for details about the rich literature.

In [15], Ma and Brunner derived a posteriori error estimates for nonlinear Volterra integro-differential equations (NVIDEs) possessing the nonstandard memory term, and studied the discontinuous Galerkin (DG) method for NVIDEs with fully discretized memory terms. The assumption of nonstandard memory term in [15] is that the derivatives of nonstandard memory term are bounded. Ma also used high order collocation methods for Black Scholes equation in economics (c.f. [16]) under the frame of previous work. Guan et al. relaxed the assumption on nonstandard memory term to Lipschitz continuity (c.f. [11]), but the Lipschitz function is constrained to small enough in some integral sense, which is little strict to extend. In [6,12], some Volterra integral equations with nonlinear convolution are also studied, and Brunner also gives some convergence and superconvergence results for Volterra functional equations with multiple proportional delays (c.f. [4]). To the best of our knowledge, there are few works about convergence of collocation methods for nonlinear convolution Volterra integral equations with multiple proportional delays, which means the memory is extended from nonlinear term to nonlinear convolution term.

Motivated by the work in [15,11], in this paper, we shall study the collocation method for nonlinear convolution Volterra integral equations with multiple delay (or: lag) functions $\theta_k = \theta_k(t),\ k = 1, 2, \dots, \overline{p}$ of the form

$$u(t) = f(t) + \sum_{k=1}^{\overline{p}} (\mathcal{V}_{\theta_k} u)(t), \quad t \in I := [0, T], \tag{1}$$

where $\overline{p}$ is some positive integer. The Volterra integral operators $\mathcal{V}_{\theta_k}(k = 1, 2, \dots, \overline{p}) : C(I) \to C(I)$ are defined by

$$(\mathcal{V}_{\theta_k} u)(t) := \int_0^{\theta_k(t)} K_k(t,s) G_1(u(t-s)) G_2(u(s)) ds,$$

* Corresponding author.
E-mail addresses: kzhang@jlu.edu.cn (K. Zhang), lijie09@mails.jlu.edu.cn (J. Li), songhm11@mails.jlu.edu.cn (H. Song).

0096-3003/$ - see front matter Crown Copyright © 2012 Published by Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2012.04.045

where f, G_1, G_2 and K_k are given smooth functions. The delay functions $\theta_k(t),\ k = 1, 2, \ldots, \bar{p}$ are assumed to have the following properties:

(1) $\theta_k(0) = 0$, and θ_k is strictly increasing on I;
(2) $\theta_k(t) \leqslant \bar{q}_k t$ on I for some $\bar{q}_k \in (0, 1)$;
(3) $\theta_k \in C^{\nu_k}(I)$ for some integer $\nu_k \geqslant 0$.

An important special case is the linear vanishing delay or proportional delay, i.e. $\theta_k(t) = q_k t = t - (1 - q_k)t := t - \tau_k(t)$ with $0 < q_k < 1$, which are known as the pantograph delay functions (c.f. [17]). We shall concern on the corresponding NCVIEMPDs given by

$$u(t) = f(t) + \sum_{k=1}^{\bar{p}} (\mathcal{V}_{q_k} u)(t) := f(t) + \mathcal{L}(G_1(u), G_2(u))(t), \quad t \in I, \tag{2}$$

where $\mathcal{L}$ is an operator which is given by

$$\mathcal{L}(G_1(u), G_2(u))(t) = \sum_{k=1}^{\bar{p}} \int_0^{q_k t} K_k(t,s) G_1(u(t-s)) G_2(u(s)) ds := \mathcal{L}u, \quad t \in I$$

and $(\mathcal{V}_{q_k} u)(t) := \int_0^{q_k t} K_k(t,s) G_1(u(t-s)) G_2(u(s)) ds$, $k = 1, 2, \ldots, \bar{p}$. Here, the nonstandard memory terms $G_1(u(t-s))G_2(u(s))$ in each $\mathcal{V}_{q_k} u$ can be different, we assume they are uniform just for simplification in the rest of this paper.

To the best of our knowledge, there exists few work on collocation method for NCVIEMPDs of form (2). In order to gain some insight approaches for nonlinear convolution Volterra integral equations, we present a study of piecewise polynomial collocation solutions for (2). There are several challenges for these NCVIEMPDs: the nonlinear convolution memory term is more tough than nonstandard memory term, and admits polynomial growth; the convergence and superconvergence proofs of integral equation are more difficult than corresponding integro-differential equations; the situations for the multiple proportional delays $\mathcal{V}_{q_k} u$ in (2) are more complex than single proportional delay.

The rest of this paper is organized as follows: In Section 2, the existence, uniqueness and regularity of the analytic solution to (2) is presented. Section 3 is devoted to construct the collocation schemes. The conditions for the uniqueness of numerical schemes, convergence and superconvergence results of collocation scheme are shown in Section 4, and in Section 5, we give some numerical experiments to verify our theoretic results. The last section is devoted to some conclusion remarks and future work.

2. Existence, uniqueness and regularity of the analytic solution

In order to show the existence, uniqueness and regularity of the analytic solution, we first introduce weighted norm space and some notations. Suppose that the space $C(I)$ is endowed with exponentially weighted norm

$$\|z\|_\sigma = \max_{t \in I} |e^{-\sigma t} z(t)|, \quad \sigma \geqslant 0,\ \forall z \in C(I)$$

and $\|v\|_\infty := \max_{t \in I} |v(t)|$. Then there exists positive number $\kappa(\sigma)$ such that

$$\kappa(\sigma)\|z\|_\infty \leqslant \|z\|_\sigma \leqslant \|z\|_\infty, \quad z \in C(I).$$

We assume the operators $\mathcal{L}$ and G_i in (2) satisfy following assumptions throughout this paper.

Assumption A: There exist positive parameters L, M, N, μ and $\lambda(\sigma)$ such that, for $i = 1, 2$ following inequalities hold

$$\begin{aligned}
&\|G_i(u_1) - G_i(u_2)\|_\sigma \leqslant M\|u_1 - u_2\|_\sigma; \quad \|G_i(u_1) - G_i(u_2)\|_\infty \leqslant M\|u_1 - u_2\|_\infty;\\
&\|G_i(u)\|_\infty \leqslant L\|u\|_\infty^\mu; \quad \|\mathcal{L}(v_1, v_2)\|_\sigma \leqslant N\|v_1\|_\sigma\|v_2\|_\sigma;\\
&\|\mathcal{L}(v_1, v_2)\|_\sigma \leqslant \lambda(\sigma)\min\{\|v_1\|_\infty\|v_2\|_\sigma, \|v_1\|_\sigma\|v_2\|_\infty\} \quad \text{with} \quad \lim_{\sigma \to \infty} \lambda(\sigma) = 0.
\end{aligned}$$

Define the linear operator $\mathcal{T} : L^\infty(I) \to L^\infty(I)$ by

$$\mathcal{T}(\varphi)(t) := f(t) + \mathcal{L}(G_1(\varphi), G_2(\varphi))(t), \quad t \in I,$$

then the Eq. (2) can be rewritten as

$$u = \mathcal{T}(u). \tag{3}$$

By using the weighted norm technique (c.f. [13,18]), we can have following theorem about the existence and uniqueness of (2) or (3).

Theorem 2.1. *Assume that the given functions and operators in (2) satisfy*

(i) $f \in C(I)$ and $K_k \in C(D_{q_k})$ with $D_{q_k} = \{(t,s) : 0 \leqslant s \leqslant q_k t\},\ k = 1, 2, \ldots, \bar{p}$;
(ii) $\mathcal{L}$ and $G_i (i = 1, 2)$ under the Assumption A.