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Finite element methods for second order linear hyperbolic interface problems

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ABSTRACT

The aim of this paper is to study finite element methods and their convergence for hyperbolic interface problems. Both semidiscrete and fully discrete schemes are analyzed. Optimal a priori error estimates in the L^2 and H^1 norms are derived for a finite element discretization where interface triangles are assumed to be curved triangles instead of straight triangles. The interfaces and boundaries of the domains are assumed to be smooth for our purpose.

1. Introduction

We consider linear hyperbolic interface problems of the form

$$u_{tt} - \nabla \cdot (a(x)\nabla u) + a_0(x)u = f(x,t) \text{ in } \Omega \times (0,T] \tag{1.1}$$

with initial and boundary conditions

$$u(x,0) = u_0, \quad u_t(x,0) = v_0 \text{ in } \Omega; \quad u(x,t) = 0 \text{ on } \partial\Omega \times (0,T] \tag{1.2}$$

and interface conditions

$$[u] = 0, \quad \left[a\frac{\partial u}{\partial \mathbf{n}}\right] = g(x,t) \text{ along } \Gamma, \tag{1.3}$$

where Ω is a bounded domain in R^2 with smooth boundary $\partial\Omega$ and $u_{tt} = \frac{\partial^2 u}{\partial t^2}$. Here, $\Omega_1 \subset \Omega$ is an open domain with C^2 smooth boundary $\Gamma = \partial\Omega_1$ and $\Omega_2 = \Omega \setminus \Omega_1$. The symbol [v] is a jump of a quantity v across the interface Γ and $\mathbf{n}$ denotes the unit outward normal to the boundary $\partial\Omega_1$. The coefficient matrix $a(x)$ is assumed to be discontinuous along Γ but piecewise smooth in each subdomain Ω_1 and Ω_2, i.e.,

$$a(x) = a^l(x) \text{ for } x \in \Omega_l, \; l = 1, 2.$$

Further, the matrix $a(x)$ is assumed to be symmetric, uniformly positive definite in Ω and $a_0(x) > 0$. Here for each $l, a^l(x)$ is a uniformly positive definite matrix. The source function f and initial functions u_0, v_0 are assumed to be sufficiently smooth.

Interface problems are often referred as differential equations with discontinuous coefficients. Hyperbolic Eqs. (1.1) with discontinuous coefficients is often used as a simple model in seismology or ocean acoustics, in which the ocean bottom is described as a multilayered fluid medium. In this case, the coefficient represents the velocity of sound which is discontinuous

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between sediment layers and u is the acoustic velocity [1]. In electromagnetism, the Eq. (1.1) corresponds to a problem in which the material occupying the interior is a dielectric rather than a metal (cf. [3]).

Several numerical schemes based on finite difference method have been designed for the approximate solutions of the hyperbolic interface problems. A general introduction on the numerical treatment for hyperbolic interface problems by means of finite difference method can be found in Le Veque's Book [16]. Three numerical schemes, namely Wendroff, TVD and WENO have been discussed in [16]. These schemes use values of the sound speed on discrete points or averaged values on grid cells. As a consequence, they do not describe accurately the position and the shape of interfaces cutting grid cells. Furthermore, due to the low regularity of the true solution the method leads to the loss in accuracy near the interface. It is then a new approach called explicit jump immersed interface method was introduced in [22]. These numerical methods ensure a given accuracy at grid points near interface, but they are difficult to implement with higher order schemes. To overcome this difficulty an explicit simplified interface method was introduced by Piraux et al. in [18] for one dimensional acoustic velocity and acoustic pressure.

The objective of this paper is to derive optimal error estimates in H^1 and L^2 norms for hyperbolic interface problems. Although a good number of articles is devoted to the finite element approximation for elliptic and parabolic interface problems [6–8,10,12], there is hardly any literature concerning the convergence of finite element solutions to the true solutions of hyperbolic interface problems. To derive $\mathbb{O}(h^m)$ $(m \geqslant 0)$ error estimates for non-interface hyperbolic problems, the finite element analysis, in general, require $u \in L^2(0,T;H^{m+1}(\Omega)) \cap H^1(0,T;H^{m-1}(\Omega)) \cap H^2(0,T;H^{m-2}(\Omega))$, see [5,13,15,19]. Because of the low global regularity of the true solution of interface problem it has been challengeable to obtain higher order convergence by means of the standard finite element error analysis technique to the interface problems (cf. [4,10,20]). In this paper, we are able to prove optimal order pointwise-in-time error estimates in L^2 and H^1 norms for the interface problem (1.1)–(1.3) if we allow interface triangles to be curved triangles. The finite element discretization is made in such a way that the grid line is isoparametrically fitted to the actual interface. Both semidiscrete and fully discrete schemes are analyzed and optimal rates of convergence are established. The key to the present analysis is the introduction of elliptic projections in each individual subdomain and the information is then transferred between the subdomains via trace.

The paper is organized as follows. In Section 2, we introduce some standard notations and recall some basic results from the literature. In Section 3, we define auxiliary projections and discuss their approximation properties. Section 4 is devoted to the error analysis for the semidiscrete finite element approximation. Finally, error estimates for the fully discrete scheme are derived in Section 5.

2. Notations and preliminaries

In this section, we shall introduce the standard notation for Sobolev spaces and norms to be used in this paper.

For $m \geqslant 0$ and real p with $1 \leqslant p \leqslant \infty$, we use $W^{m,p}(\Omega)$ to denote Sobolev space of order m with norm $\|.\|_m$ and in particular for $p = 2$, we write $W^{m,2} = H^m$. $H_0^m(\Omega)$ is a closed subspace of $H^m(\Omega)$, which is also closure of $C_0^\infty(\Omega)$ (the set of all C^∞ functions with compact support) with respect to the norm of $H^m(\Omega)$. For a fractional number s, Sobolev space H^s is defined in Adams [2]. We will be using the following equivalent definition for H^1-norm

$$\|w\|_{H^1(\Omega)} \equiv \|w\|_{H^1(\Omega_1)} + \|w\|_{H^1(\Omega_2)} \quad \forall w \in H^1(\Omega).$$

We shall also need the following spaces:

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2) \quad \text{and} \quad Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2)$$

equipped with the norms

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}$$

and

$$\|v\|_Y = \|v\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega_1)} + \|v\|_{H^1(\Omega_2)},$$

respectively.

For a given Banach space $\mathcal{B}$, we define, for $m = 0, 1$,

$$H^m(0,T;\mathcal{B}) = \left\{ u(t) \in \mathcal{B} \text{ for a.e. } t \in (0,T) \text{ and } \sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j u(t)}{\partial t^j} \right\|_{\mathcal{B}}^2 dt < \infty \right\}$$

equipped with the norm

$$\|u\|_{H^m(0,T;\mathcal{B})} = \left(\sum_{j=0}^{m} \int_0^T \left\| \frac{\partial^j u(t)}{\partial t^j} \right\|_{\mathcal{B}}^2 dt \right)^{\frac{1}{2}}.$$

We write $L^2(0,T;\mathcal{B}) = H^0(0,T;\mathcal{B})$.