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A note on the positive stable block triangular preconditioner for generalized saddle point problems

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ABSTRACT

In this paper, we give a sharp eigenvalue bound for the positive stable block triangular preconditioned matrix presented in a recent paper by Cao [Z.-H. Cao, Positive stable block triangular preconditioners for symmetric saddle point problems, Appl. Numer. Math. 57 (2007) 899–910]. The intervals containing these eigenvalues of the preconditioned matrix remove the origin, which is benefit for further study. Numerical experiments of a model Stokes problem are presented to show the estimate and the effectiveness of the positive stable block triangular preconditioners.

1. Introduction

We consider the following generalized saddle point linear system

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \quad or \quad \mathcal{A}u = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, $B \in \mathbb{R}^{m\times n}$ has full rank, $C \in \mathbb{R}^{m\times m}$ is symmetric and positive semi-definite, and $m \leqslant n$. Problem (1.1) arises in a variety of problems, such as constrained quadratic programming, constrained least squares problems, mixed finite element approximations of elliptic PDEs, computational fluid dynamics, and so on. We refer the reader to [9] for a general discussion.

When the matrix blocks $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{m\times n}$ are large and sparse, iterative methods become more attractive than direct methods for solving the saddle point problems (1.1). Many iterative methods are proposed to solve the saddle point problems (1.1), such as generalized successive overrelaxation (GSOR) method [7], parameterized inexact Uzawa methods [8,10], local Hermitian and skew-Hermitian splitting method [13] and so on. A very good survey on useful iterative methods was presented in [9]. In particular, Krylov subspace methods might be used. It is often advantageous to use a preconditioner with such iterative methods. The preconditioner should reduce the number of iterations required for convergence but not significantly increase the amount of computation required at each iteration. Preconditioning for system (1.1) has been studied in many papers, such as block triangular preconditioners [1,2,12,14,19,21], constraint preconditioners [3,15], HSS preconditioners [4,5], matrix splitting preconditioners [6,11,16,17,20] and so on.

Recently, Cao [12] studied the application of the block triangular preconditioner

$$\mathcal{P} = \begin{bmatrix} \widehat{A} & B^T \\ 0 & \widehat{C} \end{bmatrix},$$

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where $\widehat{A}$ and $\widehat{C}$ are symmetric and positive definite. Cao has showed that the preconditioned matrix $\mathcal{A}\mathcal{P}^{-1}$ is indefinite with all eigenvalues being real and the estimate for the interval containing these real eigenvalues has been studied. However, the interval containing these real eigenvalues is somewhat rough. It is necessary and very important to estimate that eigenvalues of preconditioned matrix are far from the origin for the numerical stability and convergence. This is main motivation of our work in this note. In fact, once given a preconditioner, we should ensure that the eigenvalues are away from the origin [19]. In this paper, we give a better estimate on the bounds for the eigenvalues of the preconditioned matrix $\mathcal{A}\mathcal{P}^{-1}$ under the following condition

$$\|\widehat{A}^{-1}A\|_2 \leqslant 1, \tag{1.2}$$

which is a natural assumption. In this case, the intervals containing these real eigenvalues given in this paper remove the origin. It is benefit for further study, such as the choices of the preconditioning matrices $\widehat{A}$ and $\widehat{C}$, the parameterized case [14] and so on.

The reminder of the paper is organized as follows. In Section 2, the positive stable block triangular preconditioner is recalled and a sharper bound for the eigenvalues of the preconditioned matrix than that presented in [12] is analyzed. In Section 3, numerical experiment of a model Stokes problem is presented to show the estimate and the effectiveness of the positive stable block triangular preconditioner $\mathcal{P}$.

Throughout this paper, $A > B$ means that $A - B$ is symmetric and positive definite. Let $\sigma(A)$ denote the spectrum of A. I represents the (appropriately dimensioned) identity matrix.

2. Eigenvalue analysis

We consider the eigenvalue problem

$$\mathcal{A}\mathcal{P}^{-1}\hat{u} = \lambda\hat{u}. \tag{2.1}$$

Consider additionally the block diagonal matrix

$$\mathcal{P}_0 = \begin{bmatrix} \widehat{A} & 0 \\ 0 & \widehat{C} \end{bmatrix}.$$

Then we can rewrite the eigenvalue problem (2.1) as

$$\Gamma\tilde{u} = \begin{bmatrix} \widetilde{A} & (I-\widetilde{A})\widetilde{B}^T \\ \widetilde{B} & -(\widetilde{B}\widetilde{B}^T + \widetilde{C}) \end{bmatrix}\tilde{u} = \lambda\tilde{u}, \tag{2.2}$$

where $\Gamma = (\mathcal{P}_0^{-\frac{1}{2}}\mathcal{A}\mathcal{P}_0^{-\frac{1}{2}})(\mathcal{P}_0^{-\frac{1}{2}}\mathcal{P}\mathcal{P}_0^{-\frac{1}{2}})^{-1},\ \widetilde{A} = \widehat{A}^{-\frac{1}{2}}A\widehat{A}^{-\frac{1}{2}},\ \widetilde{B} = \widehat{C}^{-\frac{1}{2}}B\widehat{A}^{-\frac{1}{2}},\ \widetilde{C} = \widehat{C}^{-\frac{1}{2}}C\widehat{C}^{-\frac{1}{2}}$ and $u = \mathcal{P}_0^{-\frac{1}{2}}\hat{u}$.

It has been studied in [12] that all the eigenvalues of the preconditioned matrix $\mathcal{A}\mathcal{P}^{-1}$ are real. We now study the bounds for the eigenvalues of $\mathcal{A}\mathcal{P}^{-1}$ or, equivalently, the bounds for the eigenvalues of Γ.

Assume first that $\tilde{A}$ has the eigenvalue 1 and X_0 is an orthogonal eigenvector basis, i.e., $\tilde{A}X_0 = X_0,\ X_0^TX_0 = I$. Let $\tilde{A} = [X_0, X]\,\mathrm{diag}(I, \Lambda)[X_0, X]^T$ be the eigenvalue decomposition of $\widetilde{A}$ with $[X_0, X]$ being orthogonal and Λ being diagonal such that its diagonal elements are less than one. Then

$$\begin{bmatrix} [X_0, X] & \\ & I \end{bmatrix}^T \Gamma \begin{bmatrix} [X_0, X] & \\ & I \end{bmatrix} = \begin{bmatrix} I & & \\ 0 & \Lambda & (I-\Lambda)X^T\widetilde{B}^T \\ \widetilde{B}X_0 & \widetilde{B}X & -(\widetilde{B}\widetilde{B}^T + \widetilde{C}) \end{bmatrix} \triangleq \begin{bmatrix} I & \\ \Gamma_2 & \Gamma_1 \end{bmatrix}, \tag{2.3}$$

where

$$\Gamma_1 = \begin{bmatrix} \Lambda & (I-\Lambda)X^T\widetilde{B}^T \\ \widetilde{B}X & -(\widetilde{B}\widetilde{B}^T + \widetilde{C}) \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 \\ \widetilde{B}X_0 \end{bmatrix}.$$

(2.3) implies that Γ has the eigenvalue 1 (may be multiple).

Without loss of generality we now assume that $1 \notin \sigma(\tilde{A})$, and $X^T\widetilde{A}X = \Lambda$, where X is orthogonal and Λ is a diagonal matrix. Under the assumption $\|\widehat{A}^{-1}A\|_2 \leqslant 1$ (cf. (1.2)) we know that $\Lambda < I$.

Consider additionally the block diagonal matrix

$$Y = \begin{bmatrix} (I-\Lambda)^{-\frac{1}{2}}X^T & \\ & I \end{bmatrix}$$

and let

$$\mathcal{H} \equiv Y\Gamma Y^{-1} = \begin{bmatrix} \Lambda & Q^T \\ Q & -S \end{bmatrix}, \tag{2.4}$$