Applied Mathematics and Computation 218 (2012) 11182–11195

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Multicolor Fourier analysis of the multigrid method for quadratic FEM discretizations

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ARTICLE INFO

Keywords:
Multigrid methods
Local Fourier analysis
Quadratic finite elements
Triangular grids
Semi-structured grids

ABSTRACT

Quadratic finite element methods offer some advantages for the numerical solution of partial differential equations (PDEs), due to their improved approximation properties in comparison to linear approaches. The algebraic linear systems arising from the discretization of PDEs by this kind of methods require an efficient resolution, and multigrid methods provide a good way to solve this problem. To design geometric multigrid methods, local Fourier analysis (LFA) is a very useful tool. However, LFA for quadratic finite element discretizations can not be performed in a standard way, since the discrete operator is defined by different stencils depending on the location of the points in the grid. In this work, a multicolor local Fourier analysis is presented to analyze multigrid solvers for this type of discretizations. With the help of this analysis, some point-wise and line-wise smoothers are analyzed. Some results showing the good correspondence between the two-grid convergence factors predicted by the analysis and the experimentally computed asymptotic convergence factors are presented. Finally, this analysis is applied to design a very efficient multigrid solver for semi-structured triangular grids, based on a hybrid-smoother.

1. Introduction

It is well-known that finite element methods (FEM) are one of the most popular techniques for numerically solving elliptic PDEs, because of their flexibility in handling unstructured meshes. For sufficiently smooth problems, a $\mathcal{O}(h^{p+1})$ convergence in the L_2-norm is achieved if finite element approximations of degree p are used. Thus, due to the improved approximation properties of higher order finite element methods in comparison to the linear case, they are often used in practical computations. Moreover, for some types of problems such as, for example, saddle point problems, high order finite element methods are required for stability reasons. For instance, the Taylor–Hood finite element method is widely used for solving the Stokes equations. These facts motivate the study of fast solvers for linear systems arising from high-order discretizations.

Multigrid methods [5,11,24,25] are among the most efficient methods for solving large algebraic systems arising from discretizations of PDEs, with optimal computational complexity, due to their ability to handle different scales present in the problem. There are two basic approaches to multigrid solvers, geometric [3,4] and algebraic multigrid, see for example [6,21] and Appendix A in [24]. In geometric multigrid, a hierarchy of grids must be defined. In algebraic multigrid, no information is used concerning the grid on which the governing PDE is discretized, and it is mainly applied to unstructured grids and to problems with large variations in the coefficients of the PDE.

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2012.05.008

To our knowledge, only a few references on multigrid methods for high-order finite elements appear in the literature, most of them related to algebraic multigrid. Many of the approaches consider polynomial approximations of order p on an initial grid, and successive coarse problems, based on discretizations of lower degree in the same grid until the polynomial order is reduced to $p = 1$, whereupon a standard multigrid algorithm is used. The p component of this algorithm was proposed in [20] and analyzed in [17] for a one-dimensional Galerkin spectral element discretization of the Laplace equation. An improved approach has recently been proposed in [22], based on a careful strategy to recover the stiffness matrix associated with the linear finite element method, from that corresponding to the high-order element discretization. Another type of approach is based on applying multigrid directly to the high-order discretization operator, see for example [13]. In the context of geometric multigrid methods, a numerical analysis of quadratic FEM is discussed in [14], where, if correct transfer-grid operators are considered, an improved approximation property is formulated enhancing the asymptotic behavior of the multigrid solver. This shows another advantage of the use of high-order finite elements, apart from that concerning the accuracy. Although the convergence of geometric multigrid methods is well-established for any conforming FEM discretization, quantitative estimates with respect to the components chosen in the algorithm are missing in the literature. This question is at the focus of this paper.

Local Fourier analysis (LFA), introduced by Brandt [3], is the most powerful tool for the quantitative analysis and design of efficient geometric multgrid methods for general problems. This analysis is based on the discrete Fourier transform, and it assumes that the problem is discretized on an infinite regular grid, neglecting boundary conditions. Moreover, this analysis is restricted to discrete linear operators with constant coefficients. From a practical point of view, LFA is helpful because it provides realistic quantitative estimates of the asymptotic multigrid convergence factor. A good introduction can be found in the paper by Stüben and Trottenberg [23] and in the books by Wesseling [25], Trottenberg et al. [24] and Wienands and Joppich [28]. Recent advances in LFA include multigrid as a preconditioner [27], for triangular meshes [7,8], optimal control problems [2], discontinuous Galerkin discretizations [12], for finite element discretizations on rectangular grids of systems of PDEs [16], and for the full multigrid method [19].

A standard local Fourier analysis cannot be directly applied to high-order finite-element discretization operators because different stencils appear depending on the location of the nodes (vertices, nodes lying on the edges and interior nodes). To overcome this trouble, a multicolor local Fourier analysis is introduced, which in some sense generalizes the two-color Fourier analysis in [15] to study the performance of red–black Gauss–Seidel for the Laplace operator. In this paper, this analysis is presented for quadratic finite element methods, but in principle, it is possible to extend the main idea to more general high-order elements. For simplicity in the presentation, we shall consider the Poisson equation as model problem, since it results in the standard benchmark problem for studying the performance of linear solvers. Therefore, let us consider the following elliptic boundary value problem

$$-\Delta u = f, \text{ in } \Omega, \tag{1}$$

$$u = g, \text{ on } \partial\Omega, \tag{2}$$

where Ω is an open bounded domain, and the corresponding algebraic linear system arising from its discretization by quadratic FEM

$$L_h u_h = f_h, \tag{3}$$

associated with a triangular partition of the domain Ω.

The organization of the paper is as follows. Section 2 is devoted to the presentation of the multicolor smoothing local Fourier analysis to study the smoothing properties of the relaxation methods. For simplicity in the development of the analysis, a triangular grid associated with a Cartesian grid is considered. Besides, a four-color smoother is introduced and analyzed by this new tool. Considering this simple triangulation, a multicolor two-grid Fourier analysis is introduced in Section 3 to estimate the asymptotic convergence rate of the multigrid method. Some Fourier results are displayed for different smoothers in Section 4. These theoretical results are confirmed by some numerical computations. In Section 5, the multicolor Fourier analysis is extended to more general regular triangulations. Finally, as application of this analysis, a multigrid solver based on hybrid smoothers is proposed for semi-structured grids in Section 6, obtaining asymptotic convergence factors around 0.1.

2. Smoothing analysis

In this section, a multicolor LFA is introduced to study the smoothing properties of relaxation schemes when quadratic FEM discretizations are considered, and such analysis is applied in detail to a four-color smoother.

2.1. Multicolor smoothing Fourier analysis

This multicolor analysis exploits the *periodicity* of the stencils and it can be seen in some sense as an extension of the two-color Fourier analysis introduced by Kuo and Levy in [15], where it was applied to study the red–black Gauss–Seidel smoother for Laplace operator.