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Common fixed point and approximation results for $\mathcal{H}$-operator pair with applications

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ABSTRACT

The existence of common fixed point results for new classes of noncommuting selfmaps satisfying generalized *I*-contraction or *I*-nonexpansive type conditions is established. As applications several invariant best approximation results are proved which unify, extend, and complement the well-known results.

1. Introduction and preliminaries

We first review needed definitions. Let M be a subset of a normed space $(X, \|.\|)$. The set $P_M(u) = \{x \in M : \|x - u\| = dist(u, M)\}$ is called the set of best approximants to $u \in X$ out of M, where $dist(u, M) = inf\{\|y - u\| : y \in M\}$. We denote by $\Im_0$ the class of closed convex subsets of X containing 0. For $M \in \Im_0$, we define $M_u = \{x \in M : \|x\| \leqslant 2\|u\|\}$. It is clear that $P_M(u) \subset M_u \in \Im_0$ [3,20,26]. We shall use N to denote the set of positive integers, $cl(S)$ to denote the closure of a set S and $wcl(S)$ to denote the weak closure of a set S. Let $f : M \to M$ be a mapping. A mapping $T : M \to M$ is called an f-contraction if, for any $x, y \in M$, there exists $0 \leqslant k < 1$ such that $\|Tx - Ty\| \leqslant k\|fx - fy\|$. If $k = 1$, then T is called f-nonexpansive. The set of fixed points of T (resp. f) is denoted by $F(T)$(resp. $F(f)$). The pair $\{f, T\}$ is called:

(1) commuting if $Tfx = fTx$ for all $x \in M$;
(2) R-weakly commuting [27] if for all $x \in M$, there exists $R > 0$ such that $\|fTx - Tfx\| \leqslant R\|fx - Tx\|$. If $R = 1$, then the maps are called weakly commuting;
(3) compatible [19] if $\lim_n\|Tfx_n - fTx_n\| = 0$ when $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n fx_n = t$ for some t in M;
(4) weakly compatible [22] if they commute at their coincidence points, i.e., if $fTx = Tfx$ whenever $fx = Tx$;
(5) occasionally weakly compatible [1,5,22] if $fTx = Tfx$ for some x with $fx = Tx$.

The set M is called q-starshaped with $q \in M$, if the segment $[q, x] = \{(1 - k)q + kx : 0 \leqslant k \leqslant 1\}$ joining q to x, is contained in M for all $x \in M$. Suppose that M is q-starshaped with $q \in F(f)$ and is both T- and f-invariant. Then T and f are called (5) C_q-commuting [2] if $fTx = Tfx$ for all $x \in C_q(f, T)$, where $C_q(f, T) = \cup\{C(f, T_k) : 0 \leqslant k \leqslant 1\}$ where $T_kx = (1 - k)q + kTx$; (6) pointwise R-subweakly commuting on M ([26]) if for given $x \in M$, there exists a real number $R > 0$ such that

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$\|fTx - Tfx\| \leqslant R dist(fx, [q, Tx])$; (7) ultraoccasionally weakly compatible ([5,21]) if I and T_k are occasionally weakly compatible for every $k \in [0, 1]$.

A map $T : M \to X$ is said to be semicompact if whenever $\{x_n\}$ is a sequence in M such that $x_n - Tx_n \to 0$ strongly, then $\{x_n\}$ has a subsequence $\{x_j\}$ in M such that $x_j \to z \in M$ strongly. Note that if $clT(M)$ is compact, then $T(M)$ is complete, $T(M)$ is bounded, and T is semicompact. The map $T : M \to X$ is said to be weakly semicompact if whenever $\{x_n\}$ is a sequence in M such that $x_n - Tx_n \to 0$ strongly, then $\{x_n\}$ has a subsequence $\{x_j\}$ in M such that $x_j \to z \in M$ weakly. The map $T : M \to X$ is said to be demiclosed at 0 if, for every sequence $\{x_n\}$ in M converging weakly to x and $\{Tx_n\}$ converges to $0 \in X$, then $0 = Tx$.

The ordered pair (f, T) of two self-maps of a metric space (X, d) is called a Banach operator pair, if the set $F(T)$ is f-invariant, namely $f(F(T)) \subseteq F(T)$. Obviously a commuting pair (f, T) is a Banach operator pair but not conversely in general, see [2,6,7,11,12,29].

In 1963, Meinardus [25] employed the Schauder fixed point theorem to prove a result regarding invariant approximation. Further generalizations of the result of Meinardus were obtained by Habiniak [9], Hicks and Humphries [10], Jungck and Sessa [23], Singh [32], Smoluk [34] and Subrahmanyam [35]. Al-Thagafi [3] extended the work of Singh [32], Smoluk [34], Subrahmanyam [35] and proved some results on invariant approximations for commuting maps. Recently, Al-Thagafi and Shahzad [4], Hussain and Jungck [15], Hussain [11,12], Hussain and Rhoades [17], Jungck and Hussain [20], O'Regan and Hussain [26], Pathak and Hussain [29,30] extended the work of Al-Thagafi [3] for pointwise R-subweakly commuting, compatible, C_q-commuting maps, Banach operator pairs and $\mathcal{P}$-operator pairs.

For more on metric fixed point theory, the reader may consult the book [24].

In this paper, we introduce two new classes of noncommuting selfmaps. The new class of *$\mathcal{H}$-operator pair* contains the classes of occasionally weakly compatible and weakly compatible selfmaps as proper subclasses and is different from the classes of *$\mathcal{P}$-operator pair* [30] and *$\mathcal{JH}$-operator pair* [13,14]. Within the class of all selfmaps I and T of a q-starshaped subset M of X, the class of ultraoccasionally weakly compatible maps is a subclass of the other new class of *$\mathcal{H}$-suboperator pair* and when additionally I is q-affine, the classes of commuting, R-weakly commuting, pointwise R-subweakly commuting, and C_q-commuting selfmaps are also subclasses of the class of *$\mathcal{H}$-suboperator pair*. For these new classes, we establish common fixed point results and obtain several invariant approximation results for convex as well as starshaped domains as applications. Our results unify, extend, and complement all the above-mentioned results.

2. $\mathcal{H}$-operator pairs, $\mathcal{JH}$-operator pairs and generalized contractions

Let X be a set and I, T selfmaps of X. A point x is called a coincidence point of I and T iff $Ix = Tx$. We shall call $w = Ix = Tx$ a point of coincidence of I and T. Let $C(I, T)$ and $PC(I, T)$ denote the sets of coincidence points and points of coincidence, respectively, of the pair (I, T).

The pair (I, T) is called *$\mathcal{P}$-operator pair* [30] if

$$d(u, Tu) \leqslant \operatorname{diam} C(I, T) \text{ for some } u \in C(I, T).$$

Clearly, occasionally weakly compatible maps I and T are $\mathcal{P}$-operators. If the self-maps I and T of X are weakly compatible, then $T(C(I, T)) \subseteq C(I, T)$, and hence I and T are $\mathcal{P}$-operators.

The pair (I, T) is called *$\mathcal{JH}$-operator pair* [14] if

$$d(u, x) \leqslant \operatorname{diam} PC(I, T) \text{ for some } x = Tu = Iu \in PC(I, T).$$

Every occasionally weakly compatible pair is an $\mathcal{JH}$-operator pair as follows;

$$\begin{aligned}
&(I, T) \text{ is an occasionally weakly compatible pair}\\
&\Rightarrow \exists v \in C(I, T) \text{ such that } ITv = TIv\\
&\Rightarrow \exists v \in C(I, T) \text{ such that } Iu = Tu \text{ for some } u = Tv = Iv \in T(C(I, T)) = PC(I, T)\\
&\Rightarrow \text{ for some } u = Tv = Iv \in PC(I, T) \text{ we have } x = Tu = Iu \in PC(I, T)\\
&\Rightarrow d(x, u) \leqslant \operatorname{diam} PC(I, T) \text{ for some } x \in PC(I, T)\\
&\Rightarrow (I, T) \text{ is an } \mathcal{JH}\text{-operator pair.}
\end{aligned}$$

The pair of mappings (I, T) is called an *$\mathcal{H}$-operator pair* if

$$\operatorname{dist}(u, I(PC(I, T))) \leqslant \operatorname{diam} PC(I, T) \text{ for some } u \in PC(I, T).$$

Every occasionally weakly compatible pair is an $\mathcal{H}$-operator pair as follows;

$$\begin{aligned}
&(I, T) \text{ is an occasionally weakly compatible pair}\\
&\Rightarrow \exists v \in C(I, T) \text{ such that } ITv = TIv\\
&\Rightarrow Iu = Tu \in IT(C(I, T)) \text{ for some } u = Tv = Iv \in T(C(I, T)) = PC(I, T) \text{ and } Tu \in PC(I, T)\\
&\Rightarrow \text{for some } u \in PC(I, T),\ Tu \in PC(I, T) \text{ and } Tu \in I(PC(I, T))\\
&\Rightarrow \text{for some } u \in PC(I, T),\ Tu \in PC(I, T) \cap I(PC(I, T))\\
&\Rightarrow \operatorname{dist}(u, I(PC(I, T)) \leqslant d(u, Tu) \text{ and } d(u, Tu) \leqslant \operatorname{diam} PC(I, T) \text{ for some } u \in PC(I, T)
\end{aligned}$$