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A modified boundary integral equation for solving the exterior Robin problem for the Helmholtz equation in three dimensions

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ABSTRACT

A boundary integral equation with modified fundamental solution to solve the exterior Robin problem for Helmholtz's equation is considered. A specific choice of the coefficients of these added terms ensure the unique solvability and, in addition, they can be chosen to ensure the minimization of the least-squares difference of the modified and the exact Green's function or the minimization of the condition number. Numerical results are reported showing the robustness and the superconvergence of the new method.

1. Introduction

Jones [6] and Ursell [10,11] introduced the theory of modifying the Green's function for the exterior Dirichlet and Neumann problem. They added radiating spherical wave functions to the fundamental solution to ensure the unique solvability of the boundary integral equation for all wave numbers. Various articles derived coefficients of these added terms to ensure different criteria. Two of them are due to Kleinman and Roach [9] and Kleinman and Kress [8], respectively. They considered minimizing the least-squares difference between the exact and modified Green's function and minimizing the condition number of the operator that solves the exterior Dirichlet and Neumann problem for a sphere and perturbations of it, respectively. We extend these ideas to derive an optimal choice for the exterior Robin problem. From the numerical point of view it is inefficient to use the full series. We show that a finite number of coefficients different from zero suffice to remove the critical wave numbers in a given range. Finally, we apply the boundary element collocation method to report numerical results for solving the exterior Robin problem for the Helmholtz equation with modified Green's function with such a choice of coefficients in three dimensions, since no numerical results have been reported yet. In addition, we present the superconvergence of the new method. Note that numerical results and superconvergence at the collocation nodes have been reported in Kleefeld [7, Chapter 4] based on the modified single-double layer approach. However, from the computational point of view this approach is very costly, since it involves composite integral operators with a weak singularity. Alternatively, one might deal directly with the hypersingularity of the normal derivative of the double layer. However, we would lose the superconvergence. A short summary and possible future work concludes this article.

2. Problem formulation

Let D be a bounded open region in $\mathbb{R}^3$ containing the origin. The boundary of D is denoted by Γ and is assumed to consist of a finite number of disjoint, closed bounded surfaces belonging to class C^2 and we assume that the complement $\mathbb{R}^3 \setminus \overline{D}$ is connected (see [4, p. 32]).

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The mathematical formulation of the exterior Robin problem consists of finding a complex-valued solution $u \in C^1(\mathbb{R}^3 \setminus D) \cap C^2(\mathbb{R}^3 \setminus \overline{D})$ solving the Helmholtz equation

$$\Delta u(A) + \kappa^2 u(A) = 0, \quad A \in \mathbb{R}^3 \setminus \overline{D}, \quad \kappa > 0$$

with the Robin boundary condition

$$\frac{\partial u}{\partial \nu}(x) + \mathrm{i}\kappa\lambda u(x) = f(x), \quad x \in \Gamma,$$

where f is a given continuous function on the surface Γ, κ is the wave number, $\lambda > 0$ (a real-valued constant) is the surface impedance (see [3]), and $u(x)$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - \mathrm{i}\kappa u\right) = 0,$$

where $r = |x|$ and the limit holds uniformly in all directions $x/|x|$. The normal derivative on the boundary exists in the sense that

$$\frac{\partial u}{\partial \nu}(x) = \lim_{h\searrow 0}\langle \nu(x),\ \mathrm{grad}\ u(x - h\nu(x))\rangle$$

exists uniformly on Γ, where ν denotes the normal directed into the exterior of D (see [4, p. 68]).

3. Integral equation

For $n = -m, \dots, m$ we denote the linearly independent spherical harmonics of order m by Y_n^m. They are given by

$$Y_n^m(\hat{x}) = \sqrt{\frac{1}{4\pi}(2n+1)\frac{(n-m)!}{(n+m)!}} P_n^m(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi},$$

where $\hat{x} = x/|x| = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in \mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ denotes a unit vector and P_n^m the associated Legendre polynomial. Further denote with j_n, y_n, and $h_n^{(1),(2)} = j_n \pm \mathrm{i}y_n$ the spherical Bessel function of order n, the spherical Neumann function of order n, and the spherical Hankel function of the first and second kind of order n, respectively. Then, the fundamental solution of the Helmholtz equation is given by

$$\Phi_\kappa(x,y) = \mathrm{i}\kappa\sum_{n=0}^{\infty}\sum_{m=-n}^{n} j_n(\kappa|x|)Y_n^m(\hat{x})h_n^{(1)}(\kappa|y|)\overline{Y_n^m(\hat{y})}, \quad |x| < |y|. \tag{1}$$

Jones [6] added a series of the form

$$\chi_\kappa(x,y) = \mathrm{i}\kappa\sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm} h_n^{(1)}(\kappa|x|)Y_n^m(\hat{x})h_n^{(1)}(\kappa|y|)\overline{Y_n^m(\hat{y})} \tag{2}$$

to Φ_κ, where the unknown coefficients a_{nm} have to satisfy certain conditions to ensure the convergence of the series (2) and its term by term derivatives. Next, define the modified fundamental solution by

$$\Psi_\kappa(x,y) = \Phi_\kappa(x,y) + \chi_\kappa(x,y).$$

With this, we define for $\sigma \in C(\Gamma)$ the compact integral operators

$$V_\kappa[\sigma](x) = \int_\Gamma \Psi_\kappa(x,y)\sigma(y)\,\mathrm{d}\Gamma_y, \quad x \in \mathbb{R}^3 \setminus \Gamma,$$
$$L_\kappa[\sigma](x) = \int_\Gamma \Psi_\kappa(x,y)\sigma(y)\,\mathrm{d}\Gamma_y, \quad x \in \Gamma,$$
$$M_\kappa^T[\sigma](x) = \int_\Gamma \frac{\partial}{\partial \nu_x}\Psi_\kappa(x,y)\sigma(y)\,\mathrm{d}\Gamma_y, \quad x \in \Gamma.$$

Assume the potential in the exterior can be written as

$$u(A) = V_\kappa[\sigma](A), \quad A \in \mathbb{R}^3 \setminus \overline{D}, \tag{3}$$

where the function $\sigma \in C(\Gamma)$ is unknown. Taking the normal derivative of (3), letting A approach $x \in \Gamma$ and using the jump relation, we obtain

$$\frac{\partial u}{\partial \nu}(x) = \left(-\frac{1}{2}\mathcal{I} + M_\kappa^T\right)[\sigma](x), \quad x \in \Gamma. \tag{4}$$