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Double precision rational approximation algorithms for the standard normal first and second order loss functions

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ABSTRACT

We present double precision algorithms based upon piecewise rational approximations for the standard normal first and second order loss functions. These functions are used frequently in inventory management. No direct approximation or closed formulation exists for the standard normal first and second order loss functions. Current state-of-the-art algorithms require intermediate computations of the cumulative normal distribution or tabulations and they do not compute to full double precision. We deal with both these issues and present direct double precision accurate algorithms which are valid in the full range of double precision floating point numbers.

1. Introduction and motivation

The normal probability distribution is used frequently in inventory management as an appropriate or simply a convenient distribution to model lead-time demand. The probability density function for the normal distribution, having an average v and a variance σ^2, is given by (1). The special case where $v = 0$ and $\sigma = 1$ is referred to as standard normal distribution. Each normal distribution can be transformed via (2) to the standard normal probability density function $\varphi(z)$ (3).

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-v)^2}{2\sigma^2}\right), \tag{1}$$

$$z_{(x)} = \frac{x-v}{\sigma}, \tag{2}$$

$$\varphi(z) = \frac{\exp(-z^2/2)}{\sqrt{2\pi}}. \tag{3}$$

The standard normal cumulative distribution Φ and its complementary function Φ_0 are given respectively by (4) and (5). The function Φ_1, given by (6), is the standard normal first order loss function or for short standard normal loss function. The standard normal second order loss function Φ_2 is defined by (7). This terminology is used by Zipkin [1,2] and will also be used in this paper. However, in literature Φ_1 is also referred to as the normal loss integral [3], or the normal linear loss integral [4]. Withers and Nadarajah [5] use the terms 'repeated integrals of the normal density function'.

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$$\Phi(z) = \int_{-\infty}^{z} \varphi(x)dx, \tag{4}$$

$$\Phi_0(z) = \int_{z}^{\infty} \varphi(x)dx = 1 - \Phi(z), \tag{5}$$

$$\Phi_1(z) = \int_{z}^{\infty} (x-z)\varphi(x)dx = \int_{z}^{\infty} \Phi_0(x)dx, \tag{6}$$

$$\Phi_2(z) = \int_{z}^{\infty} (x-z)\Phi_0(x)dx = \int_{z}^{\infty} \Phi_1(x)dx, \tag{7}$$

Within inventory management Φ_1 and Φ_2 are needed to calculate inventory level distributions, average inventory levels, ready rates, time weighted backorders and the related costs in case of normal distribution demands. As these computations are needed on a large scale, e.g. in inventory problems with a huge amount of items, one can appreciate the direct benefit of having a highly efficient and effective approximation within the range and the precision of double precision floating point numbers.

In Figs. 1 and 2 the functions Φ, Φ_0, Φ_1 and Φ_2 are respectively plotted over the z ranges $[-4,4]$ and $[0,3]$.

1.1. A brief literature review

The cumulative normal distribution Φ and its approximations have been extensively studied. There exists no closed-form expression for Φ. Abramowitz and Stegun [6] provide in Section 26.2 power series, asymptotic expansions, continued fraction expansions and polynomial and rational approximations. Waissi and Rossin [7] presented a simple sigmoid function for the approximation of the cumulative standard normal probabilities for $-8 \leqslant z \leqslant 8$. Bryc [8] presented two simple formulas

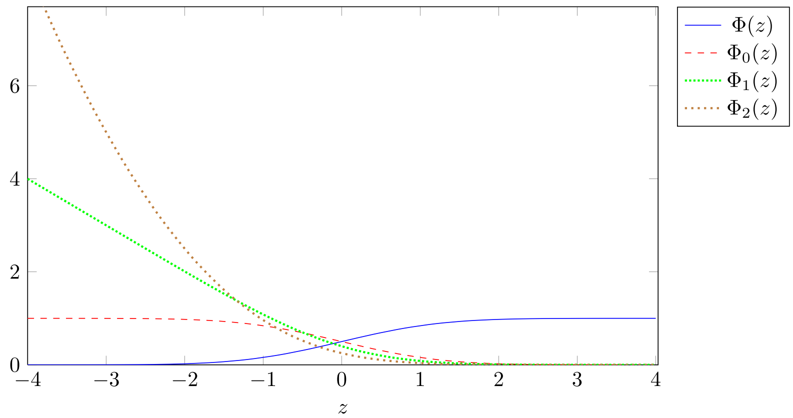

Fig. 1. Standard normal distribution and first and second loss function in range $[-4,4]$.

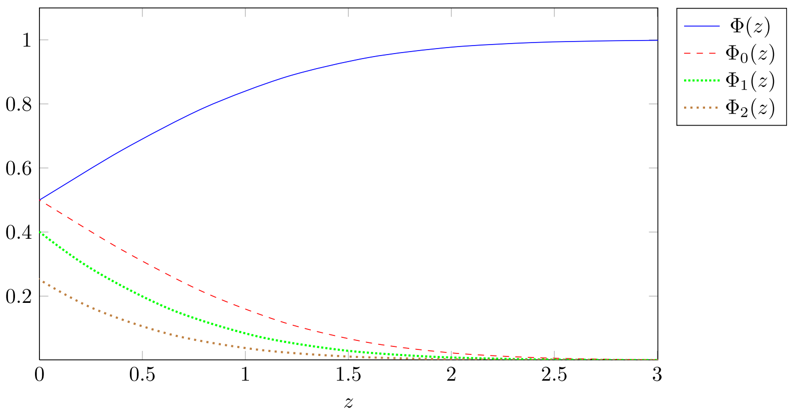

Fig. 2. Standard normal distribution and first and second loss function in range $[0,3]$.