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On a backward heat problem with time-dependent coefficient: Regularization and error estimates

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ABSTRACT

In this paper, we consider a homogeneous backward heat conduction problem which appears in some applied subjects. This problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the final data. A new regularization method is applied to formulate regularized solutions which are stably convergent to the exact ones with Holder estimates. A numerical example shows that the computational effect of the method is all satisfactory.

1. Introduction

There are several important ill-posed problems for parabolic equations. A classical example is the backward heat equation. In other words, it may be possible to specify the temperature distribution at a particular time $t < T$ from the temperature data at the final time $t = T$. This is usually referred to as the backward heat conduction problem, or the final value problem. In the present paper, we consider the problem of finding the temperature $u(x,t), (x,t) \in [0,\pi] \times [0,T]$ such that

$$u_t(x,t) = a(t)u_{xx}(x,t), \quad (x,t) \in [0,\pi] \times (0,T], \tag{1}$$
$$u(0,t) = u(\pi,t) = 0, \quad t \in [0,T], \tag{2}$$
$$u(x,T) = g(x), \quad x \in [0,\pi], \tag{3}$$

where $a(t), g(x)$ are given. The problem is called the backward heat problem, (BHP for short), the backward Cauchy problem or the final value problem. In general, the solution of the problem does not exist. Further, even if the solution existed, it would not be continuously dependent on the final data. It makes difficult to do numerical calculations. Hence, a regularization is in order.

In the special case of the problem (1)–(3) with $a(t) = 1$, the problem becomes

$$u_t(x,t) = u_{xx}(x,t), \quad (x,t) \in [0,\pi] \times (0,T], \tag{4}$$
$$u(0,t) = u(\pi,t) = 0, \quad t \in [0,T], \tag{5}$$
$$u(x,T) = g(x), \quad x \in [0,\pi]. \tag{6}$$

The problem (4)–(6) has been considered by many authors using different methods. The mollification method has been studied in [4]. An iterative algorithm with regularization techniques has been developed to approximate the BHP by Jourhmane and Mera in [10]. Kirkup and Wadsworth have given an operator-splitting method in [9]. Quasi-reversibility

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method has been used by Lattes and Lions [1], Miller [2] and the other authors [3,13,11]. The boundary element method has been also used by some authors (see [6,8]). All of them were devoted to computational aspects. However, few authors gave their error estimates from the theoretical viewpoint for the BHP except Schroter and Tautenhahn [5], Yildiz and Ozdemir [7] and Yildiz et al. [12]

Although we have many works on (4)–(6), however to the author's knowledge, so far there are few results about (1)–(3). The major object of this paper is to provide a regularization method to establish the Holder estimates. In fact, we decided to regularize the exact problem by using the form of (13) directly. It can be called the quasi-solution method (but it based on the quasi-boundary value method). By using quasi boundary value method, we have the regularized problem as follows

$$u_t(x,t) = a(t)u_{xx}(x,t), \quad (x,t) \in [0,\pi] \times (0,T], \tag{7}$$
$$u(0,t) = u(\pi,t) = 0, \quad t \in [0,T], \tag{8}$$
$$u(x,T) + \beta u(x,0) = g(x), \quad x \in [0,\pi]. \tag{9}$$

By applying the Fourier method, we can find the form of the solution of (7)–(9)

$$u(x,t) = \sum_{m=1}^{\infty} \frac{\exp\left\{-m^2 \int_0^t a(s)ds\right\}}{\beta + \exp\left\{-m^2 \int_0^T a(s)ds\right\}} g_m \sin(mx), \quad (x,t) \in [0,\pi] \times [0,T]. \tag{10}$$

The regularized solution (13) based on modifying the solution (10) of the problem (7)–(9) (noting that when $\alpha = 0$, the solution (13) is the solution (10)). In this paper, we use the regularized solution (13) directly. In Theorem 2.2, we can get the error estimate of Holder type for all t by using an appropriate parameter $\alpha \geqslant 0$. In fact, the error estimate for the case $0 < t < T$ is as follows

$$\|u(\cdot,t) - v^\epsilon(\cdot,t)\| \leqslant (1 + A_1)\epsilon^{\frac{p^2t+p\alpha}{q^2T+q\alpha}}.$$

In this case, we can choose $\alpha = 0$ and require a soft condition of the exact solution u

$$A_1 = \|u(\cdot,0)\| < \infty.$$

On the other hand, the error estimate for the case $t = 0$ is as follows

$$\|u(\cdot,t) - v^\epsilon(\cdot,t)\| \leqslant (1 + A_1)\epsilon^{\frac{p\alpha}{q^2T+q\alpha}}.$$

In order to get the the error estimate of Holder type, we choose $\alpha > 0$ and require a strong condition of the exact solution u

$$A_1 = \left(\frac{\pi}{2}\sum_{m=1}^{\infty} \exp\left\{2m^2\alpha\right\}|u_m(0)|^2\right)^{\frac{1}{2}} < \infty.$$

It requires the exact solution u is smooth enough. The remainder of the paper is divided into two sections. In Section 2, we establish the regularized solution and estimate the error between an exact solution u of problem (1)–(3) and the regularized solution u^ϵ with the Holder type. Finally, a numerical experiment will be given in Section 3.

2. Regularization and main results

We denote that $\|\cdot\|$ is the norm in $L^2(0,\pi)$. Let $\langle\cdot,\cdot\rangle$ be the inner product in $L^2(0,\pi)$ and g^ϵ be the measured data satisfying $\|g^\epsilon(\cdot) - g(\cdot)\| \leqslant \epsilon$. Let $a(t) : [0,T] \to R$ be the differentiable function for every t and satisfy

$$0 < p \leqslant a(t) \leqslant q, \quad 0 \leqslant t \leqslant T. \tag{11}$$

Suppose that Problems (1)–(3) have an exact solution u then u can be formulated as follows

$$u(x,t) = \sum_{m=1}^{\infty} \frac{\exp\left\{-m^2 \int_0^t a(s)ds\right\}}{\exp\left\{-m^2 \int_0^T a(s)ds\right\}} g_m \sin(mx), \quad (x,t) \in [0,\pi] \times [0,T]. \tag{12}$$

Let $\beta > 0$ and $\alpha \geqslant 0$, we shall approximate the solution of the backward heat problem (1)–(3) by the regularized solution as follows

$$v(x,t) = \sum_{m=1}^{\infty} \frac{\exp\left\{-m^2\left(\int_0^t a(s)ds + \alpha\right)\right\}}{\beta + \exp\left\{-m^2\left(\int_0^T a(s)ds + \alpha\right)\right\}} g_m \sin(mx), \quad (x,t) \in [0,\pi] \times [0,T]. \tag{13}$$

We note that β depends on ε such that $\lim_{\varepsilon\to 0} \beta(\varepsilon) = 0$ and α is an arbitrarily nonnegative number.

Next, we consider some lemmas which is useful to the proof of theorems.