Applied Mathematics and Computation 219 (2013) 5750–5753

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A new bound of radius with irregularity index ☆

Nihat Akgüneş *, A. Sinan Çevik

Department of Mathematics, Faculty of Science, Selçuk University, Campus 42075, Konya, Turkey

ARTICLE INFO

Keywords:
Bounds
Irregularity index
Minimum degree
Radius

ABSTRACT

In this paper, we use a technique introduced in the paper [P. Dankelmann, R.C. Entringer, Average distance, minimum degree and spanning trees, J. Graph Theory 33 (2000), 1–13] to obtain a strengthening of an old classical theorem by Erdös et al. [P. Erdös, J. Pach, R. Pollack, Z. Tuza, Radius, diameter, and minimum degree, J. Combin. Theory B 47 (1989), 73–79] on radius and minimum degree. To be more detailed, we will prove that if G is a connected graph of order n with the minimum degree δ, then the radius G does not exceed

$$\frac{3}{2}\left(\frac{n-t+1}{\delta+1}+1\right),$$

where t is the *irregularity index* (that is the number of distinct terms of the degree sequence of G) which has been recently defined in the paper [S. Mukwembi, A note on diameter and the degree sequence of a graph, Appl. Math. Lett. 25 (2012), 175–178]. We claim that our result represent the tightest bound that ever been obtained until now.

1. Introduction and preliminaries

Let $G=(V,E)$ be a simple, finite connected graph with the vertex set $V=V(G)$ of order n and the edge set $E=E(G)$. Let $d_G(x,y)$ denotes the *distance* (lengh of the shortest path) between two vertices x,y of G. The *degree* $deg_G(x)$ of a vertex x of G is the number of vertices adjacent to x. Among all degrees, the *minimum degree* of vertices is denoted by δ in graph G. We also have the term *degree sequence* $DS(G)$ which is a sequence of degrees of vertices of G. Depending on $DS(G)$, in [10], it has been recently defined a new parameter for graphs, namely the *irregularity index* of G and denoted by $t=t(G)$. In fact t is the number of distinct terms in $DS(G)$. Although there exist very huge number of studies on the degree sequence of graphs (see, for instance, [8,9] and the list of references in them), there is actually a single reference (see [10]) came through our attention about the irregularity index of G.

The *eccentricity* of a vertex v, denoted by $\epsilon(v)$, in a connected graph G is the maximum distance between v and any other vertex u of G. For a disconnected graph, all vertices are defined to have infinite eccentricity. It is well known that the *diameter*, denoted by $diam(G)$, is the maximum distance between any two vertices of G. It is quite clear that $diam(G)$ is equal to the *maximum eccentricity* among all vertices of G. On the other hand, the *minimum eccentricity* is called the *radius* of G and denoted by

$$rad(G)=\min_u\left\{\max_v\{d_G(u,v)\}\right\}.$$

Due to [12], for a connected graph G, the inequality

☆ This work is partially supported by the Scientific Research Projects Office (BAP) of Selçuk University.

* Corresponding author.
E-mail addresses: nakgunes@selcuk.edu.tr (N. Akgüneş), sinan.cevik@selcuk.edu.tr (A. Sinan Çevik).

0096-3003/$ - see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2012.11.081

$$rad(G) \leqslant diam(G) < 2\,rad(G)$$

always holds. We note that graph distance parameters have been studied extensively by several authors (see for example [1,6,7,11] and the citation in their reference lists).

By considering any connected graph of order n with the minimum degree δ ($\geqslant 2$), in the classical paper [3], Erdös et al. proved the bound

$$rad(G) \leqslant \frac{3}{2}\frac{n-3}{\delta+1} + 5, \tag{1}$$

for radius of G. Moreover, in [[5] Theorem 1], it has been recently showed that if $n \geqslant 2\delta + 2$ and $rad(G) \geqslant 3$, then

$$rad(G) \leqslant \frac{3}{2}\left\lfloor \frac{n}{\delta+1} \right\rfloor. \tag{2}$$

In this paper, by considering the parameters t, n and δ, and taking into account the results in papers [3,5], we will prove that there exists an upper bound (in Theorem 2.2 below) for the radius of any simple connected graph G. To do that we will use a technique introduced by Dankelmann and Entringer in [2]. Furthermore we will remark and show that (see Remarks 2.3 and 2.4 below) this bound is stronger than the bound given in (1) and, in some cases, the bound in (2).

2. The main result

Unless stated otherwise, G will denote a simple, connected graph of order n with the minimum degree δ.

Before presenting the main result, let us recall some other fundamentals related to the title. (We note that these materials can also be found in any graph theory textbooks, see for instance [4]).

The *neighbourhood* $N_G(x)$ of a vertex $x \in V(G)$ is the set of all vertices adjacent to x. The *closed neighbourhood* $N_G[x]$ of a vertex $x \in V(G)$ contains $N_G(x)$ and the vertex x itself. For a subset $S \subseteq V$, let us assume that $G[S]$ denotes the subgraph induced by S in G. Then the distance between a vertex x and S, denoted by $d_G(x,S)$, is defined as $\min_{v\in S} d_G(x, v)$. The closed neighbourhood of S is the set $\cup_{x\in S} N_G[x]$ and denoted by $N_G[S]$. The *kth power* of G, denoted by G^k, is the graph with the same vertex set as G in which two vertices $u \neq v \in V(G)$ are adjacent if $d_G(u, v) \leqslant k$. For a subset $A \subseteq V(G)$, the subgraph of G^k induced by A is denoted by $G^k[A]$. For a positive integer k, a *k-packing* of G is a subset $A \subseteq V(G)$ with $d_G(a,b) > k$ for all $a, b \in A$. We finally recall that, for a real number r, we denote by $\lfloor r \rfloor$ the greatest integer $\leqslant r$, and by $\lceil r \rceil$ the least integer $\geqslant r$ in our results. It is clear that

$$r \leqslant \lceil r \rceil < r + 1. \tag{3}$$

For a subset $A \subseteq V(G)$, we have the following lemma which plays a central role in the proof of our main result.

Lemma 2.1 [10]. *If A is maximal 2-packing set, then*

$$|A| \leqslant \frac{n-t+1}{\delta+1},$$

where t is the irregularity index of G.

The main theorem is the following.

Theorem 2.2. *Suppose that the irregularity index of G is t. Then we have an upper bound*

$$rad(G) \leqslant \frac{3}{2}\left(\frac{n-t+1}{\delta+1} + 1\right) \tag{4}$$

for the radius of G. In fact, this bound is essentially tight.

Proof. Assume that the degree sequence $DS(G)$ of G has t distinct terms, in other words, the irregularity index is t.

We first need to find a maximal 2-packing $A \subseteq V(G)$ of G. This can be obtained as in the following procedure: for a chosen vertex v of $V(G)$, let $A = \{v\}$. If there exists another vertex u in $V(G)$ having the condition $d_G(u,A) = 3$, add u to A. After that add all such these vertices u' having the same condition $d_G(u',A) = 3$ to A until each of the vertices not in A is within distance two of A.

As the next step, let $T_1 \leqslant G$ be the forest with the vertex set $N_G[A]$ and the edge set consists of all edges incident with a vertex in A. By the construction on A, there are $|A| - 1$ edges in G which each of them joining two neighbours of distinct elements of A, and whose addition to T_1 yields a tree $T_2 \leqslant G$. Now every vertex u not in T_2 is adjacent to some other vertex u'' in T_2. Let $T[A]$ be a spanning tree of G with the edge set $E(T_2) \cup \{uu'' | u \in V(G) - V(T_2)\}$. By the definition, since $T^3[A]$ is connected, we have

$$rad(T[A]) \leqslant 3\,rad(T^3[A]) \tag{5}$$