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On the solutions of fractional Swift Hohenberg equation with dispersion

K. Vishal [a,*], S. Das [a,b], S.H. Ong [b], P. Ghosh [c]

[a] Department of Applied Mathematics, Indian Institute of Technology (BHU), Varanasi 221 005, India
[b] Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia
[c] Department of Mechanical Engineering, Indian Institute of Technology (BHU), Varanasi 221 005, India

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ABSTRACT

In this article, the approximate solutions of the non-linear Swift Hohenberg equation with fractional time derivative in the presence of dispersive term have been obtained. The fractional derivative is described in Caputo sense. Time fractional nonlinear partial differential equations in the presence of dispersion and bifurcation parameters have been computed numerically to predict hydrodynamic fluctuations at convective instability for different particular cases and results are depicted through graphs.

1. Introduction

The Swift–Hohenberg (S–H) equation

$$\partial_t u = \mu u - \left(1 + \partial_x^2\right)^2 u + \gamma u^2 - u^3, \tag{1}$$

where μ and γ are parameters, is one of the universal models proposed by Jack Swift and Pierre Hohenberg in the year 1977 which describes the temperature and fluid velocity dynamics of the thermal convection [1]. The S–H equation which also describes the pattern formation in fluid layers confined between horizontal well-conducting boundaries is a nonlinear parabolic equation. The mathematical model for the Rayleigh–Benard convection involves the Navier–Stokes equations coupled with the transport equation for temperature. The S–H equation has important role in different branches of physics, ranging from hydrodynamics such as Taylor–Couette flow [2] to nonlinear optics and in the study of lasers [3]. The equation also plays key role in the studies of pattern formation [4]. Large time behavior of solution of Swift–Hohenberg equation has been studied by Peletier and Rottschafer [5].

In 2009, to examine the effect of additional dispersive term in Eq. (1), Burke et al. [6], considered the equation as

$$\partial_t u = \mu u - \left(1 + \partial_x^2\right)^2 u + \delta \partial_x^3 u + \gamma u^2 - u^3, \tag{2}$$

where μ, δ and γ are parameters of the equation.

Many phenomena in engineering and applied sciences can be described successfully by developing the models using fractional calculus, i.e., the theory of derivatives and integrals of non-integer order [7–10]. Thus appearances of fractional order derivatives make the study more involved and challenging. Fractional differential equations have garnered much attention since fractional order system response ultimately converges to the integer order system response. Due to its important

* Corresponding author.
E-mail address: vishal.rs.apm@itbhu.ac.in (K. Vishal).

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applications in engineering and physics, the authors are motivated to solve the following Swift–Hohenberg equation with fractional order time derivatives in the presence of dispersive term as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \mu u - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u + \delta \frac{\partial^3 u(x,t)}{\partial x^3} + 2u^2 - u^3, \quad 0 < \alpha \leqslant 1, \tag{3}$$

where μ is bifurcation parameter and δ is dispersive parameter.

Previously, modeling was mainly restricted to linear systems for which analytical treatment is tractable. But due to the advent of powerful computers and with improved computational techniques, nowadays it is possible to tackle even nonlinear problems to some extent. Nonlinearity is a phenomenon that is exhibited by most of the systems in nature and has gained increasing popularity during last few decades. Most of the nonlinear problems do not have a precise analytical solution; especially it is hard to obtain it for the fractional order nonlinear equations. So these types of equations should be solved by any approximate methods or Numerical methods. Recently, many new approaches for the solution of nonlinear differential equations have been proposed, for example, Tanh-function method [11], Jacobian elliptic function method [12], Variational iteration method [13], Adomian decomposition method [14], Homotopy analysis method [15], Homotopy perturbation method [16–18], Modified fractional decomposition method [19] and Fractional variational iteration method [20], etc. A new fractional analytical approach via modified Riemann–Liouville derivative has been given by Khan et al. [21]. Integral Transform method is a very old and powerful technique for solving linear differential equation, but if nonlinearity occurs in the problem then one cannot apply it directly. So, there is a need of amalgamation of this method with other existing methods. Khan and Wu [22] proposed a new method called homotopy perturbation transformation method (HPTM), which is a combination of homotopy perturbation method, Laplace Transform and He's polynomials [23]. The advantage of this method is its capability of combining two powerful methods for obtaining even the exact solutions for some nonlinear equations. Like homotopy perturbation method this method also does not require small parameters in the equation. Thus it overcomes the limitations of traditional perturbation techniques.

Homotopy analysis method (HAM) is also an analytical approach to get the series solution of linear and nonlinear differential equations. Like homotopy perturbation method (HPM), this method is also independent of small/large physical parameters. It also provides a simple way to ensure the convergence of the series solution. The difference of this method with the other analytical methods is that one can ensure the convergence of the series solution by choosing a proper value of convergence-control parameter. Liao has given the Comparison between the HAM and HPM in his article [24]. He showed that the HPM is only a special case of HAM. Both methods lead to very good approximation by means of only few terms, if initial guess and auxiliary linear operator are properly chosen. The difference is that HPM has to use a good enough initial guess, but this is not absolutely necessary for HAM. This method has been successfully applied to solve both integer order and fractional differential equations [25–27].

In 2010, Akyildiz et al. [28] have solved the S–H equation by homotopy analysis method in the absence of dispersion. In 2011, the S–H equation with fractional time derivative is studied by Vishal et al. [29] in the absence of dispersion. But to the best of authors' knowledge the time fractional order S–H equation with dispersion have not yet been studied by any researcher. In this article we have applied two analytical methods, first one is homotopy perturbation method in the framework of Laplace transform i.e., HPTM and other is homotopy analysis method also in the framework of Laplace transform for the solution of the fractional order S–H equation with dispersion. The beauty of this article is the introduction of new type of residual error which helps to find out the optimal values of auxiliary parameter for getting better convergence of the solution. The salient feature of the article is the graphical presentations of the effects of dispersive term, bifurcation parameter and length of the domain on the solution with time for different fractional derivative parameter for different particular cases.

2. Basic idea of homotopy perturbation transform method (HPTM)

To illustrate the basic ideas of this method, we consider the following non-linear fractional differential equation

$$D_t^\alpha u(x,t) + Ru(x,t) + Nu(x,t) = 0, \quad 0 < \alpha \leqslant 1, \tag{4}$$

with the initial condition

$$u(x,0) = f(x), \tag{5}$$

where R is the linear differential operator, N is the nonlinear differential operator and $D_t^\alpha u(x,t)$ is the Caputo fractional derivative of function $u(x,t)$ which is defined as

$$D_t^\alpha u = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(x,\tau)}{(t-\tau)^{\alpha+1-n}}\, d\tau, \quad (n-1 < \alpha \leqslant n, n \in N), \tag{6}$$

where $\Gamma(.)$ denotes the standard Gamma function.

One of the properties of Laplace transform for Caputo fractional derivative is

$$\mathrm{L}[D_t^\alpha u] = s^\alpha \mathrm{L}[u(x,t)] - \sum_{k=0}^{n-1} u^k(x,0^+)\, s^{\alpha-1-k}. \tag{7}$$