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On the $O(1/t)$ convergence rate of the parallel descent-like method and parallel splitting augmented Lagrangian method for solving a class of variational inequalities

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ABSTRACT

In this paper, we extend parallel descent-like method (PDLM) and parallel splitting augmented Lagrangian method (PSALM) for structured monotone variational inequalities whose operator is composed by three separable operators, and prove their $O(1/t)$ convergence rate.

1. Introduction

Let Ω be a closed convex set in $\mathcal{R}^n$ and F be a continuous mapping from $\mathcal{R}^n$ to itself. The variational inequality problem, denoted by $\mathrm{VI}(\Omega, \mathrm{F})$, is to find a vector $u \in \Omega$, such that

$$(u' - u)^T \mathrm{F}(u) \geqslant 0, \quad \forall u' \in \Omega. \tag{1}$$

Suppose that the variational inequality has the following separable structure:

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathrm{F}(u) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix}, \tag{2}$$

$$\Omega = \{(x, y) | x \in \mathcal{X}, y \in \mathcal{Y}, Ax + By = b\}, \tag{3}$$

where $\mathcal{X} \subset \mathcal{R}^{n_1}, \mathcal{Y} \subset \mathcal{R}^{n_2}, A \in \mathcal{R}^{m \times n_1}$, $B \in \mathcal{R}^{m \times n_2}$ are given matrices with full rank, and $b \in \mathcal{R}^m$ is a given vector, $f : \mathcal{R}^{n_1} \to \mathcal{R}^{n_1}$ and $g : \mathcal{R}^{n_2} \to \mathcal{R}^{n_2}$ are given monotone operators. Note that the solutions of the problem (1)–(3) are available when the following problem is solved: Find $w = (x, y, \lambda) \in \mathcal{W} := \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^m$ such that

$$\begin{cases} (x' - x)^T [f(x) - A^T \lambda] \geqslant 0, \\ (y' - y)^T [g(x) - B^T \lambda] \geqslant 0, \\ Ax + By - b = 0, \end{cases} \tag{4}$$

for any $w' = (x', y', \lambda') \in \mathcal{W}$, where $\lambda \in \mathcal{R}^m$ is the Lagrange multiplier associated with the linear constraint $Ax + By = b$. Problem (4) can be conveniently denoted by $\mathrm{VI}_2(\mathcal{W}, F)$, whereas

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$$(w'-w)^T F(w) \geqslant 0, \quad \forall w' \in \mathcal{W},$$

where

$$F(w) := F(x,y,\lambda) := \begin{pmatrix} f(x) - A^T\lambda \\ g(y) - B^T\lambda \\ Ax + By - b \end{pmatrix}. \tag{5}$$

The last two decades have witnessed impressive development on the alternating direction method (ADM) in the areas of variational inequalities and convex programming, see [5,7,10,15] to mention just a few. Assuming that $H \in \mathcal{R}^{m\times m}$ is a symmetric positive definite matrix and that $w^k = (x^k, y^k, \lambda^k)$ is a given iterate, then the ADM seeks the solution (x, y, λ) of the following problem in a determinate order,

$$\begin{aligned} &(x'-x)^T\left\{f(x) - A^T[\lambda^k - H(Ax + By^k - b)]\right\} \geqslant 0, \quad \forall x' \in \mathcal{X}, \\ &(y'-y)^T\left\{g(y) - B^T[\lambda^k - H(Ax + By - b)]\right\} \geqslant 0, \quad \forall y' \in \mathcal{Y}, \\ &\lambda = \lambda^k - H(Ax + By - b). \end{aligned} \tag{6}$$

In (6), solving the second subvariational inequality requires the solution of the first subvariational inequality. Hence, the alternating direction method is not eligible for parallel computing in the sense that the solutions of subvariational inequalities in (6) cannot be obtained simultaneously.

For the purpose of parallel computing, the author of [9] proposed to revise (6) as follows:

$$\begin{aligned} &(x'-x)^T\left\{f(x) - A^T[\lambda^k - H(Ax + By^k - b)]\right\} \geqslant 0, \quad \forall x' \in \mathcal{X}, \\ &(y'-y)^T\left\{g(y) - B^T[\lambda^k - H(Ax^k + By - b)]\right\} \geqslant 0, \quad \forall y' \in \mathcal{Y}, \\ &\lambda = \lambda^k - H(Ax + By - b). \end{aligned} \tag{7}$$

where the variables of the involved subvariational inequalities are not crossed and thus suitable for parallel computing. The solution of (7) is denoted by $\bar{w}^k = (\bar{x}^k, \bar{y}^k, \bar{\lambda}^k)$. To ensure convergence of the iterative sequence, the new iterate $w^{k+1} = (x^{k+1}, y^{k+1}, \lambda^{k+1})$ is generated by a descent step whose descent direction depends on w^k and $\bar{w}^k$, more specifically,

$$w^{k+1} = w^k - \alpha_k G^{-1} M(w^k - \bar{w}^k), \tag{8}$$

where

$$M = \begin{pmatrix} A^THA & & \\ & B^THB & \\ & & H^{-1} \end{pmatrix}, \tag{9}$$

G is a given proper positive-definite matrix and α_k is a judiciously chosen positive step size. We call the method parallel splitting augmented Lagrangian method (PSALM).

The recent work [14] improved the parallel splitting augmented Lagrangian method by refining the descent directions in the descent steps, rather than $-G^{-1}M(w^k - \bar{w}^k)$ as in (8). Consequently, a new parallel descent-like method (PDLM) for solving (1)–(3) is proposed. The method develops the PSALM in the sense that, for the same nominal conditions, the new iterate generated by this method is not farther form the solution set of $\mathrm{VI}_2(\mathcal{W}, F)$ than that generated by PSALM.

This parallel consideration makes particular sense where magnitude of data increases explosively and intensive computing infrastructure treating mass data (such as parallel and distributed computing facilities) becomes more and more advanced and popular, see e.g. [2,4]. Recently, He [11] has studied the $O(1/t)$ convergence rate of projection and contraction methods for variational inequalities with Lipschitz continuous monotone operators. He and Yuan [12] have provided a uniform proof to show the $O(1/t)$ convergence rate for both the original ADM and its linearized variant (known as the split inexact Uzawa method in image processing literature). The proof is based on a variational inequality approach which is novel in the literature, and it is very simple. This strategy motivated us to study the $O(1/t)$ convergence rate of PSALM [13]. In this paper, we extend PDLM and PSALM for problems with three separable operators, and we use the same strategy to prove the $O(1/t)$ convergence rate of PDLM and PSALM.

2. The proposed methods and some properties

In this section, we describe the PDLM and PSALM for problems with three separable operators and prove several properties which are useful to establish the main result. The projection under the G-norm, denoted by $P_{\mathcal{W},G}(\cdot)$, is defined as follows: