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Fractional variational homotopy perturbation iteration method and its application to a fractional diffusion equation ☆

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ABSTRACT

In this paper, we use the fractional variational homotopy perturbation iteration method (FVHPIM) with modified Riemann–Liouville derivative to solve a time-fractional diffusion equation. Using this method, a rapid convergent sequence tending to the exact solution of the equation can be obtained. To show the efficiency of the considered method, some numerical examples are presented.

1. Introduction

Recently, fractional differential equations have gained much attention due to the tremendous use in fluid mechanics, mathematical biology, electrochemistry, physics, and so on. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives, and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow [1]. Thanks to the effects of many researchers, several fractional differential equations have been investigated and solved, such as the impulsive fractional differential equations [2], the fractional advection–dispersion equation [3,4], certain types of time-fractional diffusion equation [5,6], fractional generalized Burgers' fluid [7], fractional KdV-type equations [8], space–time fractional Whitham–Broer–Kaupand equations [9], fractional heat- and wave-like equations [10], and space fractional backward Kolmogorov equation [11].

Motivated and inspired by the on-going research in this field, we will consider the following time-fractional diffusion equation

$$\frac{\partial^\alpha u(X,t)}{\partial t^\alpha} = D\Delta u(X,t) - \nabla\cdot(\mathbf{F}(X)u(X,t)), \quad 0<\alpha\leqslant 1,\ D>0, \tag{1}$$

with the initial condition

$$u(X,0)=\phi(X), \quad X\in\Omega \tag{2}$$

and boundary condition

$$u(X,t)=\varphi(X,t), \quad X\in\partial\Omega,\ t\geqslant 0. \tag{3}$$

Here, $\frac{\partial^\alpha}{\partial t^\alpha}(\cdot)$ is the modified Riemann–Liouville derivative [12–14] of order α defined in Section 2, Δ is the Laplace operator, ∇ is the Hamilton operator, $\Omega=[0,L_1]\times[0,L_2]\times\cdots\times[0,L_d]$ is the spatial domain of the problem, d is the dimension of the space, $X=(x_1,x_2,\ldots,x_d)$, $\partial\Omega$ is the boundary of Ω, $u(X,t)$ denotes the probability density function of finding a particle at X in time t, the positive constant D depends on the temperature, the friction coefficient, the universal gas constant and finally

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on the Avagadro constant, $\mathbf{F}(X)$ is the external force [15,16]. Eq. (1) can be interpreted as modeling the diffusion of a particle under the action of the external force $\mathbf{F}(X)$.

To solve the problem (1)–(3), we consider the FVHPIM in this work. The method is based on the variational iteration method (VIM) [17,18], homotopy perturbation method (HPM) [19,20] and modified Riemann–Liouville derivative proposed by G. Jumarie. The modified Riemann–Liouville derivative has the following merits. Firstly, the αth derivative of a constant is zero. Secondly, compared with the classical Caputo derivative [21], the definition of the fractional derivative is not required to satisfy higher integer-order derivative than α [22]. For its merits, G. Jumarie's modified derivative has been successfully applied to the probability calculus [23] and fractional Laplace problems [13].

This paper is organized as follows: in Section 2, some basic definitions of the fractional calculus and the algorithm of FHPIM are given. In Section 3, the application of the FVHPIM to the problem (1)–(3) is illustrated, and some numerical examples are presented. And conclusions are drawn in Section 4.

2. Fractional calculus and FVHPIM

Assume $f: R \to R, x \to f(x)$ denote a continuous (but not necessarily differentiable) function. Through the fractional Riemann–Liouville integral

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0, \tag{4}$$

the modified Riemann–Liouville derivative is defined as

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-\xi)^{n-\alpha} (f(\xi) - f(0)) d\xi, \tag{5}$$

where $n-1 \leqslant \alpha < n$ and $n \geqslant 1$.

G. Jumarie's fractional derivative of order α is defined by the limit

$$f^{(\alpha)} = \lim_{\varepsilon \to 0} \frac{\Delta^{\alpha} f(x)}{\varepsilon^{\alpha}}, \tag{6}$$

where

$$\Delta^{\alpha} f(x) = (Fw - 1)^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + (\alpha - k)\varepsilon). \tag{7}$$

Here $Fwf(x) = f(x+\varepsilon)$. The proposed modified Riemann–Liouville derivative as shown in Eq. (5) is strictly equivalent to Eq. (6) [13,22].

The integral with respect to $(dx)^{\alpha}$ is defined as the solution of the following fractional differential equation

$$dy = f(x)(dx)^{\alpha}, \quad x \geqslant 0, \ y(0) = 0, \ 0 < \alpha \leqslant 1, \tag{8}$$

which is provided by the following result [13]:

Let $f(x)$ denote a continuous function, then the solution of the Eq. (8) is defined by the equality

$$y = \int_0^x f(\xi)(d\xi)^{\alpha} = \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi, \quad 0 < \alpha \leqslant 1. \tag{9}$$

Now we give the main steps of the fractional variational homotopy perturbation iteration method as follows:

Step 1: Suppose that a nonlinear equation, say in two independent variables x and t, is given by

$$D_t^{\gamma} u(x,t) = L(u(x,t)) + N(u(x,t)) + g(x,t), \tag{10}$$

where $D_t^{\gamma}(\cdot)$ is the modified Riemann–Liouville derivative, $\gamma > 0$, L is a linear operator, N is a nonlinear operator, $u = u(x,t)$ is an unknown function, and $g(x,t)$ is the source inhomogeneous term.

Step 2: We construct the following correct functional

$$u_{k+1}(x,t) = u_k(x,t) + I_t^{\gamma}\{\lambda(\tau)(D_{\tau}^{\gamma} u_k(x,\tau) - L(u_k(x,\tau)) - N(\widetilde{u}_k(x,\tau)) - g(x,\tau))\}, \tag{11}$$

where λ is the Lagrange multiplier, which can be identified optimally via the variational theory. The subscript $k \geqslant 0$ denotes the kth approximation, the function $\widetilde{u}_k$ is considered as a restricted variation, that is $\delta \widetilde{u}_k = 0$.

Step 3: According to the VIM and HPM, we construct the following iteration formula

$$\sum_{k=0}^{\infty} q^k u_k(x,t) = u_0(x,t) + q\left\{\sum_{k=1}^{\infty} q^k u_k(x,t) + I_t^{\gamma}\left\{\lambda(\tau)\left(\sum_{k=0}^{\infty} q^k D_{\tau}^{\gamma} u_k(x,\tau) - \sum_{k=0}^{\infty} q^k L(u_k(x,\tau)) - \sum_{k=0}^{\infty} q^k N(\widetilde{u}_k(x,\tau)) - g(x,\tau)\right)\right\}\right\}, \tag{12}$$