Applied Mathematics and Computation 219 (2013) 6030–6038

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Families of third and fourth order methods for multiple roots of nonlinear equations ☆

Xiaojian Zhou [a,b], Xin Chen [a], Yongzhong Song [a,*]

[a] Jiangsu Key Laboratory for NSLSCS, Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210097, PR China
[b] School of Science, Nantong University, Nantong 226008, PR China

ARTICLE INFO

Keywords:
Nonlinear equations
Iterative method
Multiple roots
Convergence order
Optimal order

ABSTRACT

This paper presents two families of higher-order iterative methods for solving multiple roots of nonlinear equations. One is of order three and the other is of order four. The presented iterative families all require two evaluations of the function and one evaluation of its first derivative, thus the latter is of optimal order. The third-order family contains several iterative methods known already. And, different from the optimal fourth-order methods for multiple roots known already, the presented fourth-order family use the modified Newton's method as its first step. Local convergence analyses and some special cases of the presented families are given. We also carry out some numerical examples to show their performance.

1. Introduction

One of the most important and challenging problems in scientific and engineering computations is to find the solutions to a nonlinear equation $f(x) = 0$. This paper concerns iterative methods for finding the multiple root x^* of a nonlinear equation $f(x) = 0$ with multiplicity m, i.e., $f^{(i)}(x^*) = 0,\ i = 0, 1, \cdots, m-1$, and $f^{(m)}(x^*) \neq 0$.

The modified Newton's method, given by Schröder [1], is one of the most used iterative methods known to converge quadratically for multiple roots and defined by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}. \tag{1}$$

To improve the convergence of iterative methods for multiple roots, recently, some researchers have developed some iterative methods with high order of convergence [2–12]. All these methods require the knowledge of the multiplicity m.

For example, Dong has developed two third-order methods [3] as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n + \dfrac{f(y_n)}{f(y_n) - (1-\frac{1}{m})^{m-1} f(x_n)} \dfrac{f(x_n)}{f'(x_n)} \end{cases} \tag{2}$$

☆ This work is supported by the Foundation for the Authors of the National Excellent Doctoral Thesis Award of China (200720), the "333 Project" Foundation of Jiangsu Province and the Foundation for Innovative program of Jiangsu province (CXZZ11_0870).

* Corresponding author.
E-mail addresses: zxjntu@gmail.com (X. Zhou), xchen@njnu.edu.cn (X. Chen), yzsong@njnu.edu.cn (Y. Song).

0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2012.12.041

and

$$\begin{cases} y_n = x_n - \sqrt{m}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - m\left(1 - \dfrac{1}{\sqrt{m}}\right)^{1-m}\dfrac{f(y_n)}{f'(x_n)}. \end{cases} \tag{3}$$

Another third-order method, due to Victory and Neta [4], is defined by

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(x_n)}\dfrac{f(x_n) + Af(y_n)}{f(x_n) + Bf(y_n)}, \end{cases} \tag{4}$$

where $\mu = \frac{m}{m-1}$, $A = \mu^{2m} - \mu^{m+1}$, and $B = -\frac{\mu^m(m-2)(m-1)+1}{(m-1)^2}$.

All the methods mentioned above require three function evaluations, two-function and one-derivative evaluation, per iteration. Thus, according to the famous Traub's conjecture of optimal order for the method without memory [13], the optimal convergence order of them should be four, instead of three. Recently, more and more optimal fourth-order iterative methods are presented [10–12]. Different from the third-order methods above, most of these fourth-order methods require one evaluation of f and two evaluations of f'.

In [10], Sharma and Sharma present a variant of Jarratt method for multiple roots, which has fourth-order convergence and can be written as:

$$\begin{cases} y_n = x_n - \dfrac{2m}{2+m}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{m}{8}\left[(m^3 - 4m + 8) - (m+2)^2\left(\dfrac{m}{m+2}\right)^m\dfrac{f'(x_n)}{f'(y_n)} \times \left(2(m-1) - (m+2)\left(\dfrac{m}{m+2}\right)^m\dfrac{f'(x_n)}{f'(y_n)}\right)\right]\dfrac{f(x_n)}{f'(x_n)}. \end{cases} \tag{5}$$

In [11], Li et al. present six fourth-order methods with closed formulae for multiple roots of nonlinear equations. Among them, the following two methods are more efficient, since they only require one-function and two-derivative evaluation per iteration.

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - a_1\dfrac{f(x_n)}{f'(y_n)} - \dfrac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)}, \end{cases} \tag{6}$$

where

$$a_1 = -\frac{1}{2}\frac{m(m-2)(m+2)^3\left(\frac{m}{m+2}\right)^m}{(m^3 - 4m + 8)}, \quad b_1 = -\frac{(m^3 - 4m + 8)^2}{m(m^2 + 2m - 4)^3}, \quad b_2 = \frac{m^2(m^3 - 4m + 8)\left(\frac{m+2}{m}\right)^m}{(m^2 + 2m - 4)^3}$$

and

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - a_1\dfrac{f(x_n)}{f'(x_n)} - \dfrac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)}, \end{cases} \tag{7}$$

with $a_1 = -\frac{1}{2}m(m-2),\ b_1 = -\frac{1}{m},\ b_2 = \frac{1}{m}\left(\frac{2+m}{m}\right)^m$.

Very recently, we construct a more general iteration scheme for multiple roots, which is defined by [12]

$$\begin{cases} y_n = x_n - \dfrac{2m}{2+m}\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - Q\left(\dfrac{f'(y_n)}{f'(x_n)}\right)\dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{8}$$

where the function $Q(\cdot) \in C^2(\mathbb{R})$. We show that the convergence order of the family (8) is at least four, when the following equations hold:

$$Q(u) = m, \quad Q'(u) = -\frac{1}{4}m^{3-m}(2+m)^m, \quad Q''(u) = \frac{1}{4}m^4\left(\frac{m}{2+m}\right)^{-2m},$$