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Solving invariance equations involving homogeneous means with the help of computer ☆

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ABSTRACT

Given three strict means $M,N,K:\mathbb{R}_+^2\to\mathbb{R}_+$, we say that the triple (M,N,K) satisfies the *invariance equation* if

$$K(M(x,y),N(x,y))=K(x,y)\quad (x,y\in\mathbb{R}_+)$$

holds. It is well known that K is uniquely determined by M and N, and it is called the Gauss composition $M\otimes N$ of M and N.

Our aim is to solve the invariance equation when each of the means M,N,K is either a Gini or a Stolarsky mean with possibly different parameters. This implies that we have to consider six different invariance equations. With the help of the computer algebra system Maple V Release 9, which enables us to perform the tedious computations, we completely describe the general solutions of these six equations.

1. Introduction

Throughout this paper, let $\mathbb{R}_+$ and $\mathbb{N}$ denote the set of positive real and natural numbers, respectively and let $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. A two-variable continuous function $M:\mathbb{R}_+^2\to\mathbb{R}_+$ is called a *mean* on $\mathbb{R}_+$ if

$$\min(x,y)\leqslant M(x,y)\leqslant \max(x,y)\quad (x,y\in\mathbb{R}_+)\tag{1}$$

holds. If both inequalities in (1) are strict whenever $x\neq y$, then M is called a *strict mean* on $\mathbb{R}_+$.

Given three means $M,N,K:\mathbb{R}_+^2\to\mathbb{R}_+$, we say that the triple (M,N,K) satisfies the *invariance equation* if

$$K(M(x,y),N(x,y))=K(x,y)\quad (x,y\in\mathbb{R}_+)\tag{2}$$

holds. It is well known (cf. [10]) that K is uniquely determined by M and N provided that these means are strict because, in this case, for every pair of two fixed elements $x,y\in\mathbb{R}_+$, the sequence (x_n,y_n) defined by the recursion

$$x_1:=x,\quad x_{n+1}:=M(x_n,y_n),$$

$$y_1:=y,\quad y_{n+1}:=N(x_n,y_n)$$

is convergent and

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$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = K(x,y).$$

The mean K so defined is also a strict mean and it is called the Gauss composition of M and N and is denoted by $K = M \otimes N$.

The most simple example when the invariance equation holds is the well-known identity

$$\sqrt{xy} = \sqrt{\frac{x+y}{2}\cdot\frac{2xy}{x+y}} \quad (x,y\in\mathbb{R}_+),$$

that is,

$$\mathcal{G}(x,y) = \mathcal{G}(\mathcal{A}(x,y), \mathbb{H}(x,y)) \quad (x,y\in\mathbb{R}_+),$$

where $\mathcal{A}, \mathcal{G}$, and $\mathbb{H}$ stand for the two-variable arithmetic, geometric, and harmonic means, respectively. Another less trivial invariance equation is the identity

$$\mathcal{A}\otimes\mathcal{G}(x,y) = \mathcal{A}\otimes\mathcal{G}(\mathcal{A}(x,y), \mathcal{G}(x,y)) \quad (x,y\in\mathbb{R}_+),$$

where $\mathcal{A}\otimes\mathcal{G}$ denotes Gauss' *arithmetic–geometric mean* defined by

$$\mathcal{A}\otimes\mathcal{G}(x,y) = \left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{x^2\cos^2 t + y^2 \sin^2 t}}\right)^{-1} \quad (x,y\in\mathbb{R}_+).$$

The invariance equation in more general classes of means has recently been studied extensively by many authors in various papers. The invariance of the arithmetic mean $\mathcal{A}$ with respect to two quasi-arithmetic means was first investigated by Matkowski [20] under twice continuous differentiability assumptions concerning the generating functions of the quasi-arithmetic means. These regularity assumptions were weakened step-by-step by Daróczy, Maksa, and Páles in the papers [8,9], and finally this problem was completely solved assuming only continuity of the unknown functions involved [10]. The invariance equation involving three weighted quasi-arithmetic means was studied by Burai [5,6] and Jarczyk–Matkowski [18], Jarczyk [14]. The final answer (where no additional regularity assumptions are required) has been obtained in [14]. In a recent paper, we have studied the invariance of the arithmetic mean with respect to two so-called generalized quasi-arithmetic means under four times continuous differentiability assumptions [3]. The invariance of the arithmetic mean with respect to Lagrangian means was the subject of investigation of the paper [22] by Matkowski. The invariance of the arithmetic, geometric, and harmonic means with respect to the so-called Beckenbach–Gini means was studied by Matkowski in [21]. Pairs of Stolarsky means for which the geometric mean is invariant were determined by Błasińska-Lesk–Głazowska–Matkowski [7]. The invariance of the arithmetic mean with respect to further means was studied by Głazowska–Jarczyk–Matkowski [13] and Domsta–Matkowski [11]. Further generalizations were obtained in the papers [16,15,17,19].

2. Previous results

An important class of two-variable homogeneous means are the so-called Gini means (cf. Gini [12]). Given two parameters $p, q \in \mathbb{R}$, the two-variable mean $G_{p,q} : \mathbb{R}_+^2 \to \mathbb{R}_+$ is defined by the following formula

$$G_{p,q}(x,y) = \begin{cases} \left(\frac{x^p+y^p}{x^q+y^q}\right)^{\frac{1}{p-q}} & \text{for } p \neq q,\\ \exp\left(\frac{x^p\ln x + y^p \ln y}{x^p+y^p}\right) & \text{for } p = q, \end{cases}$$

for $x, y \in \mathbb{R}_+$.

The class of Gini means is a generalization of the class of power means, since taking $q = 0$, we immediately get the power (or Hölder) mean of exponent p.

The invariance equation for Gini means, i.e., the description of the set of all 6-tuples (a,b,c,d,p,q) such that

$$G_{p,q}\big(G_{a,b}(x,y), G_{c,d}(x,y)\big) = G_{p,q}(x,y) \quad (x,y\in\mathbb{R}_+) \tag{3}$$

be valid, has been solved in the paper [2] by the authors. The main result is contained in the following theorem:

Theorem G. *Let $a,b,c,d,p,q \in \mathbb{R}$. Then the invariance Eq. (3) is satisfied if and only if one of the following possibilities hold:*

(i) $a+b = c+d = p+q = 0$, *i.e., all the three means are equal to the geometric mean,*

(ii) $\{a,b\} = \{c,d\} = \{p,q\}$, *i.e., all the three means are equal to each other,*

(iii) $\{a,b\} = \{-c,-d\}$ *and* $p+q = 0$, *i.e.,* $G_{p,q}$ *is the geometric mean and* $G_{a,b} = G_{-c,-d}$,

(iv) *there exist* $u, v \in \mathbb{R}$ *such that* $\{a,b\} = \{u+v, v\}, \{c,d\} = \{u-v, -v\}$, *and* $\{p,q\} = \{u, 0\}$ *(in this case,* $G_{p,q}$ *is a power mean),*

(v) *there exists* $w \in \mathbb{R}$ *such that* $\{a,b\} = \{3w, w\}, c+d = 0$, *and* $\{p,q\} = \{2w, 0\}$ *(in this case,* $G_{p,q}$ *is a power mean and* $G_{c,d}$ *is the geometric mean),*