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Numerical solutions of fractional Riccati type differential equations by means of the Bernstein polynomials

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ABSTRACT

In this paper, a collocation method based on the Bernstein polynomials is presented for the fractional Riccati type differential equations. By writing $t \to t^{\alpha}$ $(0 < \alpha < 1)$ in the truncated Bernstein series, the truncated fractional Bernstein series is obtained and then it is transformed into the matrix form. By using Caputo fractional derivative, the matrix forms of the fractional derivatives are constructed for the truncated fractional Bernstein series. We convert each term of the problem to the matrix form by means of the truncated fractional Bernstein series. By using the collocation points, we have the basic matrix equation which corresponds to a system of nonlinear algebraic equations. Lastly, a new system of nonlinear algebraic equations is obtained by using the matrix forms of the conditions and the basic matrix equation. The solution of this system gives the approximate solution for the truncated limited N. Error analysis included the residual error estimation and the upper bound of the absolute errors is introduced for this method. The technique and the error analysis are applied to some problems to demonstrate the validity and applicability of the proposed method.

1. Introduction

Fractional differential equations are encountered in model problems in fluid flow, viscoelasticity, finance, engineering, and other areas of applications [1–8].

In this paper, we consider the fractional Riccati type differential equation

$$\sum_{k=1}^{m} P_k(t)\frac{d^{k\alpha}y(t)}{dt^{k\alpha}} = A(t) + B(t)y + C(t)y^2, \quad m-1 < m\alpha \leqslant m, \quad 0 \leqslant t \leqslant R < \infty, \tag{1}$$

under the mixed conditions

$$\sum_{k=0}^{m-1} a_{pk}y^{(k\alpha)}(0) + b_{pk}y^{(k\alpha)}(b) = \beta_p, \quad p = 0,1,\ldots,m-1, 0 < b \leqslant R. \tag{2}$$

Here, $y(t)$ is an unknown function; $A(t)$, $B(t)$ and $C(t)$ are the functions defined in $[0,R]$; a_{pk}, b_{pk} and β_p are appropriate constants and $k\alpha$ is a constant describing the order of the fractional derivative. For $\alpha \in \mathbb{Z}^{+}$, the problem becomes a classical Riccati differential equation.

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In recent years, the fractional Riccati differential equations have been solved by the homotopy perturbation method [9], the enhanced homotopy perturbation method [9], the modified homotopy perturbation method [10], the homotopy analysis method [11] and the Adomian's decomposition method [12].

During the last decade, the fractional differential equations have been solved by means of the numerical and analytical methods such as the Adomian's decomposition method [13,14], the He's variational iteration method [15], the Taylor polynomials method [16,17], the Jacobi operational matrix method [18], the homotopy perturbation method [19], the homotopy analysis method [20], the interpolation functions [21], the operational matrix method based on the Legendre polynomials [22], the second kind Chebyshev wavelet method [23], the Bessel collocation method [24] and the Tau method [25].

On the other hand, Bhatti and Bracken [26] solved the differential equations by using the Galerkin method based on the Bernstein polynomial basis, Yousefi and Behroozifar [27] presented an operational matrix method based on Bernstein polynomials for the differential equations, Işık et al. [28,29] have studied on the Bernstein polynomial solutions of the linear pantograph equations and linear integro-differential equations with weakly singular kernel and also Işık et al. [30] have solved the high-order initial and boundary values problems by using a rational approximation based on Bernstein polynomials.

The remainder of the paper is organized as follows: The basic definitions are given in fractional calculus in Section 2. In Section 3, the Bernstein polynomials and their some properties are presented. We summarize the method in Section 4. In Section 5, the method is defined for approximate solution of the fractional problem (1), (2). In Section 6, the error analysis technique based on the residual function is developed for the present method. In Section 7, we apply the proposed method to the some problems and report our numerical finding. We end the paper with few concluding remarks in Section 8.

2. Basic definitions

In this section, we first give some basic definitions and some properties of fractional calculus in [31–36].

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty]$. Clearly, $C_\mu \subset C_\beta$ if $\beta < \mu$.

Definition 2.2. A function $f(x)$, $x > 0$, is said to be in the space $C_\mu^m,\ \mu,\ m \in \mathbb{N} \cup \{\mathbb{0}\}$, if $f^{(m)} \in C_\mu$.

Definition 2.3. The Riemann–Liouville fractional integral operator of order $\alpha \geqslant 0$ of a function, $f \in C_\mu$, $\mu \geqslant -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha \geqslant 0, \quad x > 0,$$

$$J^0 f(x) = f(x).$$

The properties of the operator J^α can be found in [31,32]; we mention only the following.
For $f \in C_\mu,\ \mu \geqslant -1,\ \alpha,\ \beta \geqslant 0$ and $\gamma > -1$:

(i) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,
(ii) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$,
(iii) $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena using fractional differential equations. Therefore, we will introduce a modified fractional differential operator D_*^α proposed by Caputo's work on the theory of viscoelasticity [33].

Definition 2.4. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt,$$

for $m - 1 < \alpha < m$, $m \in N$, $x > 0$, $f \in C_{-1}^m$ where $D = \frac{d}{dt}$.

For the Caputo derivative we have [34,35]

$$D_*^\alpha c = 0, \ (c \text{ is a constant}),$$

$$D_*^\alpha x^\beta = \begin{cases} 0, & \text{for} \quad \beta \in \mathbb{N}_0 \quad \text{and} \quad \beta < \lceil \alpha \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for} \quad \beta \in \mathbb{N}_0 \quad \text{and} \quad \beta \geqslant \lceil \alpha \rceil \text{ or } \beta \notin \mathbb{N} \quad \text{and} \quad \beta > \lfloor \alpha \rfloor. \end{cases} \tag{3}$$

We note that the approximate solutions will be found by using the Caputo fractional derivative and its properties in this study.