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Generalized composition operators on weighted Hardy spaces

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ABSTRACT

The boundedness of recently introduced generalized composition operators on weighted Hardy spaces is characterized.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$, and $H^\infty(\mathbb{D})$ the space of all bounded analytic functions on $\mathbb{D}$ with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)|$.

For $a \in \mathbb{D}$, let σ_a be the involutive Möbius transformation of the unit disk, interchanging points a and 0, that is,

$$\sigma_a(z) = \frac{a-z}{1-\bar{a}z}.$$

Let ω be a positive continuous integrable function on $[0,1)$. If such a function is extended for $z \in \mathbb{D}$ by $\omega(z) = \omega(|z|)$, it is called a *weight* function. We say, that a weight ω is *almost standard* if it is non-increasing and such that $\omega(r)/(1-r)^{1+\gamma}$ is nondecreasing for some $\gamma > 0$. By H_ω we denote the weighted Hardy space, a normed space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|^2_{H_\omega} = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < \infty, \tag{1}$$

where $dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} r dr d\theta$ stands for the normalized area measure on $\mathbb{D}$ (for this and some related spaces see [1,5]). By some calculation we see that a function $f(z) = \sum_{n=0}^\infty a_n z^n$ belongs to H_ω if and only if

$$\sum_{n=0}^\infty \omega_n |a_n|^2 < \infty,$$

where $\omega_0 = 1$ and

$$\omega_n = 2n^2 \int_0^1 r^{2n-1} \omega(r) dr, \quad n \in \mathbb{N}.$$

The sequence $(\omega_n)_{n\in\mathbb{N}_0}$ is called the *weight sequence* of the weighted Hardy space.

For a Borel measure μ, we define the following Dirichlet-type space

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$$\mathcal{D}_\mu(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|^2_{\mathcal{D}_\mu} := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 d\mu(z) < \infty \right\}.$$

Let $g \in H(\mathbb{D})$ and φ be a holomorphic self-maps of $\mathbb{D}$. The next operator denoted by $J_{g,\varphi}$ was introduced in [8,16] (for closely related n-dimensional operators see also [20,24])

$$J_{g,\varphi}f(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)d\zeta, \quad f \in H(\mathbb{D}). \tag{2}$$

It is called the *generalized composition operator*. The operator $J_{g,\varphi}$ is a generalization of the integral-type operator J_g, which is obtained for $\varphi(z) = z$.

When $g(z) = \varphi'(z)$, then $J_{g,\varphi}$ is reduced to the difference of a composition operator and a point evaluation operator, more precisely $J_{\varphi',\varphi} = C_\varphi - \delta_{\varphi(0)}$. Operator (2) is one of products of linear operators on $H(\mathbb{D})$, which have attracted some attention recently, mainly due to the fact that these kind of operators make a link between classical function theory and operator theory. For some results in the area see, e.g. [2–40] and the references therein.

Given two Banach spaces Y and Z, we recall that a linear map $T : Y \to Z$ is bounded if $T(E) \subset Z$ is bounded for every bounded subset E of Y.

Here we characterize the boundedness of the generalized composition operator $J_{g,\varphi}$ on the weighted Hardy space.

Throughout the paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $BC \leqslant A \leqslant CB$.

2. Boundedness of $J_{g,\varphi}$ on H_w

In this section, we characterize the boundedness of generalized composition operators on weighted Hardy spaces.

Recall that for $a, z \in \mathbb{D}$, the pseudohyperbolic distance d between a and z is defined by

$$d(a,z) = |\sigma_a(z)|.$$

For $r \in (0,1)$ and $a \in \mathbb{D}$, denote by $D(a,r)$, the pseudohyperbolic disk whose pseudohyperbolic center is a and whose pseudohyperbolic radius is r, that is

$$D(a,r) = \{z \in \mathbb{D} : d(a,z) < r\}.$$

Since σ_a is a linear fractional transformation, the pseudohyperbolic disk $D(a,r)$ is also a Euclidean disk. Except for the special case $D(0,r) = r\mathbb{D} = \{z : |z| < r\}$, the Euclidean center and Euclidean radius of $D(a,r)$ do not coincide with the pseudohyperbolic center and pseudohyperbolic radius. The Euclidean center and Euclidean radius of $D(a,r)$ are respectively

$$\frac{1-r^2}{1-r^2|a|^2}a \quad \text{and} \quad \frac{1-|a|^2}{1-r^2|a|^2}r.$$

For each $r \in (0,1)$, there is a positive integer M and a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{D}$ with positive separation constant, that is,

$$\inf_{n\neq m} |\sigma_{z_n}(z_m)| > 0,$$

such that $\cup_{n=1}^\infty D(z_n, r) = \mathbb{D}$ and every point in $\mathbb{D}$ belongs to at most M sets in the family $\{D(z_n, 3r)\}_{n\in\mathbb{N}}$.

The notation $A(D(a,r))$ will denote the area of $D(a,r)$. It is well-known that

$$A(D(a,r)) \asymp |1-\bar{a}z|^2 \asymp (1-|a|^2)^2 \asymp (1-|z|^2)^2 \asymp A(D(z,r)) \tag{3}$$

for every $z \in D(a,r)$.

In what follows, we make use of Carleson measure techniques, so we give a short introduction to Carleson sets and Carleson measures first.

Let I be an arc of $\partial\mathbb{D}$ and let $S(I)$ be the Carleson windows defined by

$$S(I) = \{z \in \mathbb{D} : 1-|I| \leqslant |z| < 1,\ z/|z| \in I\},$$

where $|I|$ denotes the Lebesgue measure of I.

Let $0 < \alpha < \infty$. Recall that a positive Borel measure μ on $\mathbb{D}$ is called α-Carleson measure if

$$\|\mu\|_\alpha = \sup_{|I|>0} \frac{\mu(S(I))}{|I|^\alpha} < \infty.$$

In the next result, we characterize a Carleson-type positive Borel measures μ on weighted Hardy spaces.

Before we formulate and prove the result we quote an auxiliary result from Ref. [5].