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On the ★-Sylvester equation $AX \pm X^{\star} B^{\star} = C$

Chun-Yueh Chiang [a,*], Eric King-Wah Chu [b], Wen-Wei Lin [c]

[a] Center for General Education, National Formosa University, Huwei 632, Taiwan
[b] School of Mathematical Sciences, Building 28, Monash University, VIC 3800, Australia
[c] Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan

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ABSTRACT

We consider the solution of the ★-Sylvester equations $AX \pm X^{\star}B^{\star} = C$, for $\star = T$, H and $A, B, \in \mathbb{C}^{n\times n}$, and the related linear matrix equations $AXB^{\star} \pm X^{\star} = C$, $AXB^{\star} \pm CX^{\star}D^{\star} = E$ and $AX \pm X^{\star}A^{\star} = C$. Solvability conditions and numerical methods are considered, in terms of the (generalized and periodic) Schur and QR decompositions. We emphasize the square cases where $m = n$ but the rectangular cases will be considered.

1. Introduction

In [4], the Lyapunov-like linear matrix equation

$$A^{\star}X + X^{\star}A = B, \quad A, X \in \mathbb{C}^{m\times n} \ (m \neq n)$$

with $(\cdot)^{\star} = (\cdot)^T$ was considered using generalized inverses. Applications occur in Hamiltonian mechanics. At the end of [4], the more general Sylvester-like equation

$$A^{\star}X + X^{\star}C = B, \quad A, C, X \in \mathbb{C}^{m\times n} \ (m \neq n)$$

was proposed without solution. The equation (with $\star = T$) was studied, again using generalized inverses, in [11,16]. However in [16], the necessary and sufficient conditions for solvability may be too complicated for most applications. The formula for X for the special case, assuming $m = n$, $B^T = B$ and the invertibility of $A \pm C^T$, may not be numerically stable or efficient (see Appendix B for the main result). In [11], some necessary or sufficient conditions for solvability were derived. A (seemingly wrong) formula for X in terms of generalized inverse was also proposed (see Section 2.2 for more details on the approach taken in [11]). Consult also [5, Lemma 5.10] and [18, Lemma 7], where solvability conditions for the ★-Sylvester equations with $m = n$ were obtained, without considering the details of the solution process. In recent years, an extensive amount of iterative methods based on the conjugate gradient method were studied and developed for solving the generalized T-Sylvester equation

$$AXB + CX^TD = E, \quad A, B, C, D, E, X \in \mathbb{R}^{n\times n}.$$

See, e.g., [15,21–23] and the references cited therein.

In this paper, the (numerical) solution of the ★-Sylvester equation (with $\star = T, H$; the latter indicating the complex conjugate transpose), as well as some related equations, will be studied. Our tools include the (generalized and periodic) Schur,

* Corresponding author.
E-mail addresses: chiang@nfu.edu.tw (C.-Y. Chiang), eric.chu@monash.edu (E.K.-W. Chu), wwlin@am.nctu.edu.tw (W.-W. Lin).

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singular value and QR decompositions [13]. We are mainly interested in the square cases when $m = n$. Other relative work can be found in [1,19,20].

Our interest in the $\star$-Sylvester equation originates from the solution of the $\star$-Riccati equation

$$XAX^{\star} + XB + CX^{\star} + D = 0$$

from an application related to the palindromic eigenvalue problem [5–7,18] (where eigenvalues appears in reciprocal pairs λ and $\lambda^{-\star}$). The solution of the $\star$-Riccati equation is difficult and the application of Newton's method is an obvious possibility. The solution of the $\star$-Sylvester equation is required in the Newton iterative process. Interestingly, the $\star$-Sylvester and $\star$-Lyapunov equations behave very differently from the ordinary Sylvester and Lyapunov equations. For example, from Theorem 2.1 below, the $\star$-Sylvester equation is uniquely solvable only if the generalized spectrum $\sigma(A, B)$ (the set of ordered pairs $\{(a_i, b_i)\}$ representing the eigenvalues of the matrix pencil $A - \lambda B$ or matrix pair (A, B) by $\lambda_i = a_i/b_i$) does not contain λ and $\lambda^{-\star}$ simultaneously, some sort of *apalindromic*[1] requirement. For more details of this application, see Appendix A.

The paper is organized as follows. After this introduction, Section 2 considers the $\star$-Sylvester equation, in terms of its solvability, the proposed algorithms and the associated error analysis. Section 3 contains several small illustrative examples. Section 4 considers some generalizations of the $\star$-Sylvester equation—$AXB^{\star} \pm X^{\star} = C$, $AXB^{\star} \pm CX^{\star}D^{\star} = E$ and the $\star$-Lyapunov equation $AX \pm X^{\star}A^{\star} = C$. (Similar equations like $AX \pm BX^{\star} = C$ can be treated similarly and will not be pursued here.) We conclude in Section 5 before describing two applications (in addition to those in [5,6,18]) and a solution formula in terms of generalized inverse from [16] in the Appendices.

2. $\star$-Sylvester equation

Consider the $\star$-Sylvester equation

$$AX \pm X^{\star}B^{\star} = C, \quad A, B, X \in \mathbb{C}^{n\times n}. \tag{2.1}$$

This includes the special cases of the T-Sylvester equation when $\star = T$ and the H-Sylvester equation when $\star = H$. Justified by associated applications and for efficient exposition, we shall consider $\star = H, T$ simultaneously, as far as possible.

Remark 2.1. Although it is seemingly simpler to consider only the "+" case in (2.1) and replace B by it negative for the "−" case, this will not be applicable for $\star$-Lyapunov equations. Also, note that some solvability conditions are dependent on the sign while others are not, thus our results will be more revealing with ± in (2.1). While all these features make our results more general, the (small) price to pay will be the occasional confusing symbols to unfamiliar eyes. If necessary, please concentrate on one of the four cases, e.g. the $\star = T$ and "−" case, which interests you most.

With the Kronecker product and $\star = T$, (2.1) can be written as

$$\mathcal{P}\mathrm{vec}(X) = \mathrm{vec}(C), \mathcal{P} \equiv I \otimes A \pm (B \otimes I)E, \tag{2.2}$$

where vec$X$ stacks the columns of X into a column vector and E is the permutation matrix which maps vec (X) into vec (X^T) [2]; i.e., $E = \sum_{1\leqslant i,j\leqslant n} e_j e_i^T \otimes e_i e_j^T$, where e_i denotes the i-th column of the $n \times n$ identity matrix I_n. The matrix operator on the left-hand-side of (2.2) is $n^2 \times n^2$ and the application of Gaussian elimination and the like will be inefficient. In addition, the approach ignores the structure of the original problem, introducing errors to the solution process unnecessarily.

For the $\star = H$ case, (2.1) can be rewritten as an expanded T-Sylvester equation:

$$\mathcal{A}\mathcal{X} \pm \mathcal{X}^T\mathcal{B}^T = \mathcal{C}, \quad \mathcal{A}, \mathcal{B}, \mathcal{X} \in \mathbb{R}^{2n\times 2n},$$

where

$$\mathcal{A} \equiv \begin{bmatrix} A_r & A_i \\ -A_i & A_r \end{bmatrix}, \quad \mathcal{B} \equiv \begin{bmatrix} B_r & B_i \\ -B_i & B_r \end{bmatrix}, \quad \mathcal{C} \equiv \begin{bmatrix} C_r & C_i \\ -C_i & C_r \end{bmatrix}, \quad \mathcal{X} \equiv \begin{bmatrix} X_r & X_i \\ -X_i & X_r \end{bmatrix};$$

with the original matrices written in their real and imaginary parts:

$$A = A_r + iA_i, \quad B = B_r + iB_i, \quad C = C_r + iC_i, \quad X = X_r + iX_i.$$

The Kronecker product formulation for T-Sylvester equations can then be applied. Such a formulation will be less efficient for the numerical solution of (2.1), but may be useful as a theoretical tool.

A more efficient approach will be to transform (2.1) by some unitary P and Q, so that (2.1) becomes:

$$PAQ \cdot \overline{Q}^T XP^T \pm PX^T\overline{Q} \cdot Q^T B^T P^T = PCP^T \tag{2.3}$$

or, for $\star = H$:

$$PAQ \cdot Q^H XP^H \pm PX^H Q \cdot Q^H B^H P^H = PCP^H. \tag{2.4}$$

[1] Not being palindromic, with "anti-palindromic" already describes something different.