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A generalization of the local Hermitian and skew-Hermitian splitting iteration methods for the non-Hermitian saddle point problems ☆

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ABSTRACT

For large sparse saddle point problems, Jiang and Cao studied a class of local Hermitian and skew-Hermitian splitting (LHSS) iteration methods (see M.-Q. Jiang, Y. Cao, On local Hermitian and skew-Hermitian splitting iteration methods for generalized saddle point problems, J. Comput. Appl. Math. 231 (2009) 973–982). In this paper, we generalized these methods and propose a class of generalized local Hermitian and skew-Hermitian splitting (GLHSS) iteration schemes for solving the non-Hermitian saddle point problems. We derive conditions for guaranteeing the convergence of these iterative methods. With different choices of the parameter matrices, the generalized iterative methods lead to a series of existing and new iterative methods. Numerical experiments for a model Stokes problem are provided, further show that the new iteration methods are feasible and effective.

1. Introduction

We consider the solution of systems of linear equations of the block 2×2 form:

$$\begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \quad \text{or} \quad \mathscr{A}u = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is non-Hermitian matrix and its Hermitian part $H = \frac{1}{2}(A + A^*)$ is positive definite matrix, $B \in \mathbb{C}^{m\times n}$ is a matrix of full row rank, $x, f \in \mathbb{C}^n, y, g \in \mathbb{C}^m$, and $m \leqslant n$. These assumptions guarantee the existence and uniqueness of the solution of linear systems (1.1); see [1–5]. The linear system (1.1) is called the non-Hermitian saddle point problems. We further assume that the matrices A and B are large and sparse; see [6–8].

The system (1.1) arise in a variety of scientific computing and engineering applications, including computational fluid dynamics, constrained and weighted least-squares problem, constrained optimization, image reconstruction and registration, parameter identification problems, mixed finite element approximations of elliptic PDEs and Stokes problems, and so on; see [9,6,8] and references therein.

A large variety of methods for solving linear systems of the form (1.1) can be found in the literature, including direct and iterative methods. For example, Uzawa methods [4,2,10–14], preconditioned Krylov subspace iteration methods [15,8,16–18], Hermitian and skew-Hermitian splitting (HSS) method as well as its accelerated variants [19–21,3,22,23], and restrictively preconditioned conjugate gradient methods [15,24–26]. We refer to a comprehensive survey [8] for algebraic properties and iterative methods for saddle point problems.

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In this paper, we shall focus on the numerical iterative solution for the non-Hermitian saddle point problem and assume that the Hermitian part $H = \frac{1}{2}(A + A^*)$ of the non-Hermitian matrix A is dominant, and get a generalized local Hermitian and skew-Hermitian splitting (GLHSS) method for the non-Hermitian saddle point problem. By choosing the involved parameters and parameter matrices, we can recover many known iteration methods and obtain many new ones. The convergence of these methods are also discussed in depth.

The paper is organized as follows. In Section 2, we propose the GLHSS method for non-Hermitian saddle point problems, and deduce the condition for guaranteeing its convergence. In Section 3, we derive several algorithms by different choices of the parameter matrices. Numerical experiments for a model Stokes problem are presented in Section 4, to shown the feasibility and effectiveness of new methods. Finally, in Section 5 we draw some conclusions.

2. Generalized iteration method

We consider the following special matrix splitting:

$$\begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} = \begin{bmatrix} Q_1 + H & 0 \\ -B + Q_3 & Q_2 \end{bmatrix} - \begin{bmatrix} Q_1 - S & -B^* \\ Q_3 & Q_2 \end{bmatrix},$$

where $H = \frac{1}{2}(A + A^*), S = \frac{1}{2}(A - A^*)$ are the Hermitian and the skew-Hermitian parts of A, respectively. The matrix B still has full row-rank, $Q_1 \in \mathbb{C}^{n\times n}$ is a Hermitian positive semi-definite matrix, $Q_2 \in \mathbb{C}^{m\times m}$ is a Hermitian positive definite matrix, and $Q_3 \in \mathbb{C}^{(m\times n)}$ is arbitrary matrix. Then we have the following GLHSS iteration scheme for solving the non-Hermitian saddle point problems (1.1):

$$\begin{bmatrix} Q_1 + H & 0 \\ -B + Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} Q_1 - S & -B^* \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} f \\ g \end{bmatrix}.$$

The corresponding computational process is described below.

Algorithm 2.1. GLHSS iteration method

$$\begin{cases} x_{n+1} = x_n + (Q_1 + H)^{-1}(f - Ax_n - B^* y_n), \\ y_{n+1} = y_n + Q_2^{-1}((B - Q_3)x_{n+1} + Q_3 x_n + g). \end{cases} \tag{2.1}$$

It is evident that the iteration method given above is a more generalized case of the local Hermitian and skew-Hermitian splitting (LHSS) and the modified LHSS (MLHSS) iteration methods used in [23]. Thus, the LHSS and MLHSS methods are both special cases of the GLHSS method.

In the following, we deduce the convergence of the GLHSS iteration method. Note that the iteration matrix of the GLHSS iteration method is:

$$\Gamma = \begin{bmatrix} Q_1 + H & 0 \\ -B + Q_3 & Q_2 \end{bmatrix}^{-1} \begin{bmatrix} Q_1 - S & -B^* \\ Q_3 & Q_2 \end{bmatrix}. \tag{2.2}$$

Let $\rho(\Gamma)$ denote the spectral radius of the iterative matrix Γ. Then the GLHSS iterative scheme (2.1) converges if and only if $\rho(\Gamma) < 1$, see [4,2,27]. Let λ be an eigenvalue of Γ and $(u^*, v^*)^*$ be its corresponding eigenvector, where $u \in \mathbb{C}^n$ and $v \in \mathbb{C}^m$. Then we have:

$$\begin{bmatrix} Q_1 - S & -B^* \\ Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} Q_1 + H & 0 \\ -B + Q_3 & Q_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}, \tag{2.3}$$

or equivalently,

$$\begin{cases} (\lambda H + S + (\lambda - 1)Q_1)u + B^* v = 0, \\ ((1 - \lambda)Q_3 + \lambda B)u + (1 - \lambda)Q_2 v = 0. \end{cases} \tag{2.4}$$

To obtain a convergence condition, we first give some lemmas to be used later.

Lemma 2.1 [2]. *Both roots of the complex quadratic equation $\lambda^2 + \phi\lambda + \varphi = 0$ have modulus less than one if and only if $|\phi - \bar{\phi}\varphi| + |\varphi|^2 < 1$, where $\bar{\phi}$ denotes the conjugate complex of ϕ.*

Lemma 2.2 [23]. *If S is a skew-Hermitian matrix, then $\mathbf{i} \cdot S$ ($\mathbf{i}$ is the imaginary unit) is a Hermitian matrix and u^*Su is a purely imaginary number or zero for all $u \in \mathbb{C}^n$. In particular, if S is a skew-symmetric matrix, then $u^*Su = 0$ for all $u \in \mathbb{C}^n$.*

Lemma 2.3. *Let A be a non-Hermitian matrix with the positive definite Hermitian part $H = \frac{1}{2}(A + A^*)$, and the matrix B have full row-rank. Let the matrix Γ be defined as in (2.2). If λ is an eigenvalue of Γ, then $\lambda \neq 1$.*