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Implicit one-dimensional discrete maps and their connection with existence problem of chaotic dynamics in 3-D systems of differential equations

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ABSTRACT

New types of chaotic attractors for some 3-D autonomous systems of ordinary quadratic differential equations are founded. Examples are given.

1. Introduction

Chaos, as a very interesting complex nonlinear phenomenon, has been intensively studied in the last four decades in the science, mathematics and engineering communities. Recently, the chaos has been found to be very useful and having a great potential in many technological disciplines, such as information and computer sciences, power systems protection, biomedical systems analysis, flow dynamics and liquid mixing, encryption and communications, and so on. It is not surprising, therefore, that academic researches on chaotic dynamics has evolved from traditional trends of analyzing and understanding chaos to new directions of controlling and utilizing it (see, for example Refs. [1–13] and many references cited therein).

However there are only a few publications in which (from the mathematical point of view) the existence of chaotic dynamics is rigorously proved. It is known that a numerical evidence may occasionally be misleading, since computer simulations have finite precision and experimental measurements have finite ranges in the time or frequency domain. The witnessed behavior may be an artifact of the observation device due to physical limitations. Thus, a rigorous proof is often necessary for full understanding of the chaotic dynamics in various nonlinear dynamic systems.

The present work is a continuation of the article [2]. Its appearance is dictated by the desire to generalize results derived in [2] and, simultaneously, to do these results more rigorous.

The general theory of n-dimensional implicit discrete mappings was represented in [14]. It was based on a study of the special explicit positive and negative mappings.

In the present paper for the analysis of 1-dimensional implicit discrete mappings, generated by the Ricker discrete population model [2], other approach is offered. A contingency proof between the Ricker mapping [2] and some 1-dimensional explicit discrete mapping with the known chaotic properties is basis of this approach.

We will consider the following system (see [2]):

$$\begin{cases} \dot{x}(t) = a_1x(t) + a_{11}y^2(t) + a_{12}y(t)z(t) + a_{22}z^2(t), \\ \dot{y}(t) = b_1y(t) + c_1z(t) + bx(t)y(t), \\ \dot{z}(t) = -c_1y(t) + b_1z(t) + cx(t)z(t). \end{cases} \tag{1}$$

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By $Disc = a_{12}^2 - 4a_{11}a_{22}$ denote the discriminant of the quadratic form $a_{11}y^2 + a_{12}yz + a_{22}z^2$. For system (1) it is necessary to distinguish two cases: (1) $Disc > 0$ and (2) $Disc \leqslant 0$.

(1) Let $Disc > 0$. In this case we have $a_{11}y^2 + a_{12}yz + a_{22}z^2 = (\alpha y + \beta z)(\gamma y + \delta z)$, where $\alpha, \beta, \gamma, \delta$ are real known constants. Introduce new variables u and v defined by the formulas $u = \alpha y + \beta z$, $v = \gamma y + \delta z$. Assume that $\Delta = \alpha\delta - \beta\gamma \neq 0$. Then in these variables system (1) will take the form

$$\begin{cases} \dot{x}(t) = a_1 x(t) + u(t)v(t), \\ \dot{u}(t) = r_1 u(t) + p_1 v(t) + x(t)(r_2 u(t) + p_2 v(t)), \\ \dot{v}(t) = q_1 u(t) + h_1 v(t) + x(t)(q_2 u(t) + h_2 v(t)). \end{cases} \tag{2}$$

where $r_1 = (b_1\delta - c_1\gamma)/\Delta$, $p_1 = (-b_1\beta + c_1\alpha)/\Delta$, $r_2 = b\delta/\Delta$, $p_2 = -b\beta/\Delta$, $q_1 = (-c_1\delta - b_1\gamma)/\Delta$, $h_1 = (c_1\beta + b_1\alpha)/\Delta$, $q_2 = -c\gamma/\Delta$, $h_2 = c\alpha/\Delta$.

Suppose that in system (2) $r_2 = h_2 = 0$. (It can be if $\alpha = \delta = 0$). Then we obtain a family of the systems similar to families which were investigated previously in numerous works (see, for example [15–21]).

For example in [16] the system

$$\begin{cases} \dot{x}(t) = ax(t) - y(t)z(t), \\ \dot{y}(t) = -by(t) + x(t)z(t), \\ \dot{z}(t) = mx(t) - cz(t) + x(t)y(t) \end{cases} \tag{3}$$

was considered. In this three-dimensional autonomous system with simple system structure a new chaotic 2×2-scroll attractor was generated.

In [17–19] the system

$$\begin{cases} \dot{x}(t) = -(ab/(a+b))x(t) - y(t)z(t) - c, \\ \dot{y}(t) = ay(t) + x(t)z(t), \\ \dot{z}(t) = bz(t) + x(t)y(t) \end{cases} \tag{4}$$

was analyzed. For system (4) the existence of two homoclinic orbits was proved. It was shown that system (4) has a chaotic attractor of homoclinic type, when parameters of the system satisfy some conditions.

At last in [4] has proposed the new 3-D autonomous quadratic system

$$\begin{cases} \dot{x}(t) = a(y(t) - x(t)) + ey(t)z(t), \\ \dot{y}(t) = cx(t) + dy(t) - x(t)z(t), \\ \dot{z}(t) = -bz(t) + x(t)y(t), \end{cases} \tag{5}$$

which can generate a four-wing chaotic attractor with complicated topological structures.

Thus, systems (3)–(5) are special cases of system (1).

(2) Now let $Disc \leqslant 0$. As far as we know this case was first considered in [2]. We mark that results got in [2] are incomplete. Therefore this case is the theme of further researches and in the present work it is not considered.
Further it will be shown that with the exception of known attractors, which were found in [2], system (1) possesses chaotic attractors are not indicated in [2]. The existence of these attractors is explained by a presence in the dynamic of system (1) an implicit iterated process of the Ricker type [2].

2. Research of implicit function $F(x,y) = y \cdot \exp(\alpha \cdot y) - \lambda \cdot x \cdot \exp(\beta \cdot x) = 0$

(A) From the beginning in this section a few basic concepts of one-dimensional maps will be introduced.

By definition, put $\mathbb{V} = [0, \infty)$. Introduce on the space $\mathbb{V}$ a metric d by the rule: $\forall v_1, v_2 \in \mathbb{V} d = \|v_1 - v_2\|$. Define by $h : \mathbb{V} \to \mathbb{V}$ an explicit function under the formula $h(v) = v \cdot \exp(r - v)$, where $r > 0, v \in \mathbb{V}$.

Let $v^* \neq 0$ be a fixed point of h. It is obvious that $v^* = r$.

Denote by $B_\delta(a) = \{v \in \mathbb{V} : d(v, a) < \delta\}$ and $\overline{B_\delta}(a) = \{v \in \mathbb{V} : d(v, a) \leqslant \delta\}$ open and closed balls in $\mathbb{V}$. It is clear that $a \geqslant \delta \geqslant 0$.

Definition 1 [12]. A point $v^* \in \mathbb{V}$ is called an expanding fixed point of h in $\overline{B_\delta}(v^*)$ for some constant $\delta > 0$, if $h(v^*) = v^*$ and there exists a constant $\lambda > 1$ such that

$$d(h(x), h(y)) \geqslant \lambda d(x, y), \quad \forall\, x, y \in \overline{B_\delta}(v^*).$$

Furthermore, v^* is called a regular expanding fixed point of h in $\overline{B_\delta}(v^*)$ if v^* is an interior point of $h(B_\delta(v^*))$.