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Positive solutions of semi-positone Hammerstein integral equations and applications ☆

Zengqin Zhao

College of Mathematics Sciences, Qufu Normal University, Qufu, Shandong 273165, PR China

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ABSTRACT

Existence of solutions for semi-positone integral equations of Hammerstein type is obtained by using the fixed point index method. Many boundary value problems of differential equations can be transformed into these integral equations. A specific example is given.

1. Introduction

There are many researches on integral equations of Hammerstein type, which can be used to study differential equations with a variety of boundary conditions arising in the applied fields (see [1–3] and their references). The theory of semi-positone integral equations has been emerging as an important area of investigations in recent years (see [4–6]).

In this paper, we consider the semi-positone integral equation of Hammerstein type

$$u(x) = \int_G k(x,y)f(y,u(y))dy, \tag{1.1}$$

where G is a compact set in R^n of positive measure. We will work in the space $C(G)$ of continuous functions endowed with the usual supremum norm. Under reasonable semi-positone conditions, we obtain the existence of the positive solutions for the Eq. (1.1). We prove that Green's functions of many boundary value problems for differential equations satisfy the conditions of the integral kernel $k(x,y)$. Finally, we give a specific example.

Let $R^+ = [0,+\infty)$. By the semi-positone case of (1.1), we mean that $f(x,u)$ may change sign and there exists a nonnegative function $q(x)$ such that $f(x,u) + q(x) \geqslant 0$, for $a.e.$ $(x,u) \in G \times R^+$. A function $w(x)$ is called a positive solution of the problem (1.1) if $w(x)$ is a solution of (3.1) and $w(x) \geqslant 0$, $w(x) \not\equiv 0$ on G.

In [5,6], the nonlinearity is $q(t)m(t,u)$ with $m(t,u) \geqslant -\eta$, thus, $q(t)m(t,u) \geqslant -\eta q(t)$. The nonlinearity $f(t,u)$ satisfies $f(x,u) \geqslant -q(x)$ in this paper. It is unnecessary that the nonlinearity is a product of two functions. Our integral equations are discussed in a compact set of R^n, which are more general than those in an interval of R^1 in [5,6]. The conditions we use here differ from those in the majority of papers that we know and the methods of proof are different from those in [5,6] in essence.

For the sake of convenience, we formulate some conditions.

(H1) The kernel $k : G \times G \to R^+$ is measurable, and for every $\tau \in G$ we have

$$\lim_{x\to\tau} |k(x,y) - k(\tau,y)| = 0, \quad \text{for } a.e.\ y \in G.$$

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E-mail addresses: zqzhaoy@163.com, zqzhao@mail.qfnu.edu.cn

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(H2) There exist a continuous function $a : G \to R^+$, a measurable function $b : G \to R^+$ and a positive number c such that

$$c\,a(x)\,b(y) \leqslant k(x,y) \leqslant a(x), \quad k(x,y) \leqslant b(y), \quad \forall x, y \in G.$$

(H3) The non-linearity $f(x,u) : G \times R^+ \to R^1$ satisfies Caratheodory conditions; that is, $f(\cdot, u)$ is measurable for each fixed $u \geqslant 0$ and $f(x,\cdot)$ is continuous for a.e. $x \in G$; and there exist functions $p(x), q(x) \in L(G, R^+),\ h \in C(G \times R^+, R^+)$ such that

$$0 \leqslant f(x,u) + q(x) \leqslant p(x)\,h(x,u) \quad \forall (x,u) \in G \times R^+.$$

(H4) There exists a compact subset $G_0 \subset G$ with measure $(G_0) > 0$ such that

$$a(x) > 0 \quad \forall x \in G_0, \quad k(x,y) > 0, \quad \forall (x,y) \in G_0 \times G_0,$$

$$\lim_{u \to +\infty} \frac{f(x,u) + q(x)}{u} = +\infty, \quad \text{uniformly on } G_0.$$

(H5) There exists $r_0 > 0$ such that

$$\int_G q(x)dx < c\,r_0, \quad \int_G b(x)p(x)dx < \frac{r_0}{\sup\{h(x,u) | x \in G,\, 0 \leqslant u \leqslant r_0\}}.$$

We have the following theorem.

Theorem 1.1. *Suppose that the conditions (H1)–(H5) are satisfied, Then the Eq. (1.1) has a positive solution $w(x)$, and there exist constants $M_1 > m_1 > 0$ such that*

$$m_1\, a(x) \leqslant w(x) \leqslant M_1\, a(x), \quad x \in G. \tag{1.2}$$

In order to prove the main results, we need the following lemma which is obtained in [7].

Lemma 1.1. *Let E be a real Banach space, θ denote the zero element of E. Ω be a bounded open subset of E, with $\theta \in \Omega$ and $A : \overline{\Omega} \cap Q \to Q$ is a completely continuous operator, where Q is a cone in E.*

(i) *Suppose that $Au \ \ \mu u,\ \forall u \in \partial\Omega \cap Q,\ \mu \geqslant 1$, then the fixed point index $i(A, \Omega \cap Q, Q) = 1$.*
(ii) *Suppose that $Au \not\leqslant u$ (that is u-$Au \neq \in Q$), $\forall u \in \partial\Omega \cap Q$, then $i(A, \Omega \cap Q, Q) = 0$.*

The theory of cones and fixed point theorems for mappings in ordered Banach spaces can be seen in [7,8].

2. Proof of Theorem 1.1

Let $E = C(G),\ \|\cdot\|$ denote the sup norm of the Banach space E,

$$\Omega_r = \{u \in E \mid \|u\| < r\}, \quad P = \{u \in E \mid u(x) \geqslant 0, x \in G\},$$

$$P_1 = \{u \in E \mid u(x) \geqslant c\|u\|a(x), x \in G\}, \tag{2.1}$$

where $a(x)$ and c are as in (H2). Clearly, P_1 is a cone in E, (E, P_1) is an ordered Banach spaces.

We write

$$\phi(r) = \sup\{h(x,u) |\ x \in G,\ 0 \leqslant u \leqslant r\}, \quad [u(x)]^+ = \max\{u(x),\ 0\},$$

$$u_0(x) = \int_G k(x,y)q(y)dy, \quad x \in G \tag{2.2}$$

and define the operator A by

$$Au(x) = \int_G k(x,y)\big(f(y, [u(y) - u_0(y)]^+) + q(y)\big)dy,\ \forall u(x) \in P. \tag{2.3}$$

From (H2) and (H5) it is known that $u_0(x)$ makes sense, $u_0(x) \in P$ and

$$u_0(x) \leqslant a(x)\int_G q(y)dy = \frac{c\|u\|a(x)}{c\|u\|}\int_G q(y)dy \leqslant \frac{u(x)}{c\|u\|}\int_G q(y)dy, \quad \forall u \in P_1 \setminus \{\theta\}. \tag{2.4}$$

Since $[u(x) - u_0(x)]^+ \leqslant u(x) \leqslant \|u\|$ for any $u(x) \in P$, it follows from (H3) that

$$f(x, [u(x) - u_0(x)]^+) + q(x) \leqslant p(x)\,h(x, [u(x) - u_0(x)]^+) \leqslant p(x)\phi(\|u\|), \quad \forall x \in G. \tag{2.5}$$