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Bounded superposition operators between weighted Banach spaces of analytic functions

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ABSTRACT

We characterize all entire functions that transform a weighted Banach spaces of analytic functions $\mathcal{H}_{\mu_1}^{\infty}$ into another space of the same kind $\mathcal{H}_{\mu_2}^{\infty}$ by superposition for very general weights μ_1 and μ_2.

1. Introduction

A function μ defined on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ is called *a function weight* if it is continuous, strictly positive and bounded on $\mathbb{D}$. Given a weight μ, we denote by $\mathcal{H}_{\mu}^{\infty}$ the set of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mu} := \sup_{z\in\mathbb{D}} \mu(z)|f(z)| < \infty. \tag{1}$$

It is known that $\mathcal{H}_{\mu}^{\infty}$ is a Banach space with the norm which is defined in (1), therein called *the weighted Banach spaces of analytic functions*. The space $\mathcal{H}_{\mu}^{\infty}$ is connected with the study of growth conditions of analytic functions and was studied in detail in [3,4]. It is clear that when $\mu(z) = 1$ for all $z \in \mathbb{D}$ we get the space of all analytic bounded functions on $\mathbb{D}$. We denote this particular function space by $\mathcal{H}^{\infty}$. Also, when $\mu(z) = \left(1-|z|^2\right)^{\alpha}$, with $\alpha > 0$, we get the Korenblum spaces $\mathcal{H}^{-\alpha}$. These spaces have been studied extensively by many authors (see [16,12,10]). The growth spaces are closely related to many spaces of analytic functions; for example, it is known that the Bergman spaces A^p with $p > 1$ (see [12,10] for the definition and properties of the Bergman spaces) is contained in $\mathcal{H}^{-\frac{2}{p}}$, and $\mathcal{H}^{-\beta} \subset A^p$ for every $\beta < \frac{1}{p}$. Also, the α-Bloch spaces $\mathcal{B}^{\alpha}$ with $\alpha > 1$ coincide with $\mathcal{H}^{-(\alpha-1)}$ (see Proposition 4.1).

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two metric subspaces of $H(\mathbb{D})$, and let ϕ be a complex-valued function in the plane such that $\phi \circ f$, belongs to $\mathcal{H}_2$, for all $f \in \mathcal{H}_1$. Then we say that ϕ, acts by *superposition* from $\mathcal{H}_1$ into $\mathcal{H}_2$. The *superposition operator* $S_{\phi} : \mathcal{H}_1 \to \mathcal{H}_2$ with symbol ϕ is defined by $S_{\phi}(f) := \phi \circ f$. Observe that if $\mathcal{H}_1$ contains the linear functions and S_{ϕ} maps $\mathcal{H}_1$ into $\mathcal{H}_2$, then ϕ must be an entire function. Thus, the natural questions here are:

For which entire functions ϕ do we have $S_{\phi}(\mathcal{H}_1) \subset \mathcal{H}_2$? When is S_{ϕ} bounded?

Here, an operator (possibly non-linear) acting between two metric spaces is said to be *bounded* if it maps bounded sets into bounded sets.

Analogous problems in the context of real variables have an increasingly extensive history (see [2]), but the study of such natural questions on spaces of analytic functions has developed rather recently. The operators S_{ϕ} that map one Bergman space into another or into the area Nevalinna class were characterized in terms of their symbols in [8]. The results of Cámera and Giménez have been extended by other authors to other spaces of analytic functions, where it is remarkable the works of

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Vukotić et. al. in [1,5–7]. It should be noted that in this last references the authors were able to relate the problem of superposition with the univalent interpolation in several spaces of analytic functions (see [7]), where a very interesting geometric construction of simply connected domain is given. Also, they could establish properties of the superposition operators in certain spaces of analytic functions in terms of the order and type of the entire functions (see [6]). The techniques developed by Vukotić et. al. have been used by Xiong in [13] to study when the superposition operator maps Q_p spaces into α-Bloch spaces with $p > 0$ and $\alpha \in (0,1)$, by Xu in [14] to establish properties of the superposition operators mapping an α_1-Bloch space into another α_2-Bloch spaces with $\alpha_1, \alpha_2 > 0$ and by Castillo, Ramos-Fernández and Salazar in [9] to characterize all entire functions that transform a Bloch-Orlicz space into the α-Bloch space by superposition. Recently, in [11], Girela and Márquez characterize the superposition operators which apply Q_s spaces into H_p. It should be mentioned that in that work, Girela and Márquez make use of the John–Nirenberg theorem to significantly modify some of the arguments used by Vukotić et. al.

The purpose of this note is to show that we can find certain maximal functions in the weighted Banach spaces of analytic functions which allow us to characterize bounded superposition operators that maps one space $\mathcal{H}^\infty_{\mu_1}$ into another space of the same kind $\mathcal{H}^\infty_{\mu_2}$ for very general weights μ_1 and μ_2 (see Theorems 3.1, 3.2, 3.3). As a consequence of our results, we characterize superposition operators that map one α-Bloch space with $\alpha > 1$ into another of the same class (see [14]).

This paper is organized as follows: In Section 2, we recall the definition of the associated weight, and we show the existence of a certain maximal function in $\mathcal{H}^\infty_\mu$, where the norm of the evaluation functional is obtained. In Section 3, we give conditions on the weights μ_1 and μ_2 to characterize the superposition operators S_ϕ that take the space $\mathcal{H}^\infty_{\mu_1}$ to the space $\mathcal{H}^\infty_{\mu_2}$. Finally, in Section 4, we use the results obtained in Section 3 to characterize the entire functions ϕ that map the α-Bloch space with $\alpha > 1$ into another space of the same kind. These results were obtained earlier by Xu in [14].

2. The associated weight

From the definition of the space $\mathcal{H}^\infty_\mu$, where the weight μ is fixed, we can deduce that given $z \in \mathbb{D}$, there exists a constant $K_{z,\mu}$, depending only on z and μ, such that

$$|f(z)| \leqslant K_{z,\mu}\|f\|_{H^\infty_\mu}$$

for all $f \in H^\infty_\mu$. This means that the evaluation functional δ_z at z is continuous on H^∞_μ, and we can define the *associated weight* with μ, denoted as $\widetilde{\mu}$, by

$$\widetilde{\mu}(z) = \frac{1}{\|\delta_z\|} = \frac{1}{\sup\left\{|f(z)| : \|f\|_{H^\infty_\mu} \leqslant 1\right\}},$$

where $z \in \mathbb{D}$, and $\|\delta_z\|$ denotes the norm of the evaluation functional at z. In [3], it was shown that $\widetilde{\mu}$ satisfies the following properties:

(1) $\widetilde{\mu}$ is a weight and $0 < \mu(z) < \widetilde{\mu}(z)$ for all $z \in \mathbb{D}$,

(2) H^∞_μ is isometrically equal to $H^\infty_{\widetilde{\mu}}$ and $\|f\|_{H^\infty_\mu} = \|f\|_{H^\infty_{\widetilde{\mu}}}$ for all $f \in H^\infty_\mu$.

Also we have the following very useful property, for which we include a short proof due to its importance in the proofs of our results.

Lemma 2.1. *For every $z \in \mathbb{D}$, there exists $f_z \in H^\infty_\mu$ such that $\|f_z\|_{H^\infty_\mu} \leqslant 1$ and*

$$\widetilde{\mu}(z)|f_z(z)| = 1,$$

Proof. Let $z \in \mathbb{D}$ be fixed. By definition of the associated weight $\widetilde{\mu}(z)$, we can find a sequence $\{f_n\} \subset H(\mathbb{D})$ such that $\|f_n\|_{H^\infty_\mu} \leqslant 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty}|f_n(z)| = \frac{1}{\widetilde{\mu}(z)}.$$

Next, we invoke Montel's theorem to show that the family $\{f_n\}$ is normal since this sequence is uniformly bounded on compact subsets of $\mathbb{D}$. Thus we can suppose that f_n converges uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function f. In particular, we have $|f(z)| = \lim_{n\to\infty}|f_n(z)| = \frac{1}{\widetilde{\mu}(z)}$. Finally, for any $w \in \mathbb{D}$, we have

$$\widetilde{\mu}(w)|f(w)| = \lim_{n\to\infty}\widetilde{\mu}(w)|f_n(w)| \leqslant \lim_{n\to\infty}\|f_n\|_{H^\infty_\mu} \leqslant 1,$$

which implies that $f \in \mathcal{H}^\infty_\mu$ and $\|f\|_{\mathcal{H}^\infty_\mu} \leqslant 1$. The result follows with $f_z = f$.

A weight μ is called *essential* if there exists a constant $C > 0$ such that

$$\widetilde{\mu}(z) \leqslant C\mu(z) \tag{2}$$