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Identities of symmetry for Bernoulli polynomials arising from quotients of Volkenborn integrals invariant under S_3

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ABSTRACT

In this paper, we derive eight basic identities of symmetry in three variables related to Bernoulli polynomials and power sums. These and most of their corollaries are new, since there have been results only about identities of symmetry in two variables. These abundance of symmetries shed new light even on the existing identities so as to yield some further interesting ones. The derivations of identities are based on the p-adic integral expression of the generating function for the Bernoulli polynomials and the quotient of integrals that can be expressed as the exponential generating function for the power sums.

1. Introduction and preliminaries

Let p be a fixed prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. For a uniformly differentiable (also called continuously differentiable) function $f : \mathbb{Z}_p \to \mathbb{C}_p$ (cf. [4]), the Volkenborn integral of f is defined by

$$\int_{\mathbb{Z}_p} f(z)d\mu(z) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} f(j).$$

Then it is easy to see that

$$\int_{\mathbb{Z}_p} f(z+1)d\mu(z) = \int_{\mathbb{Z}_p} f(z)d\mu(z) + f'(0). \tag{1}$$

Let $|\cdot|_p$ be the normalized absolute value of $\mathbb{C}_p$, such that $|p|_p = \frac{1}{p}$, and let

$$E = \{t \in \mathbb{C}_p \,|\, |t|_p < p^{-\frac{1}{p-1}}\}. \tag{2}$$

Then, for each fixed $t \in E$, the function $f(z) = e^{zt}$ is analytic on $\mathbb{Z}_p$ and by applying (1) to this f, we get the p-adic integral expression of the generating function for Bernoulli numbers B_n:

$$\int_{\mathbb{Z}_p} e^{zt}d\mu(z) = \frac{t}{e^t - 1} = \sum\nolimits_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (t \in E). \tag{3}$$

So we have the following p-adic integral expression of the generating function for the Bernoulli polynomials $B_n(x)$:

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$$\int_{\mathbb{Z}_p} e^{(x+z)t} d\mu(z) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (t \in E, \ x \in \mathbb{Z}_p). \tag{4}$$

Here and throughout this paper, we will have many instances to be able to interchange integral and infinite sum. That is justified by Proposition 55.4 in [4]. Let $S_k(n)$ denote the kth power sum of the first $n+1$ nonnegative integers, namely

$$S_k(n) = \sum_{i=0}^{n} i^k = 0^k + 1^k + \cdots + n^k. \tag{5}$$

In particular,

$$S_0(n) = n+1, \quad S_k(0) = \begin{cases} 1, & \text{for } k = 0, \\ 0, & \text{for } k > 0. \end{cases} \tag{6}$$

From (3) and (5), one easily derives the following identities: for $w \in \mathbb{Z}_{>0}$,

$$\frac{w \int_{\mathbb{Z}_p} e^{xt} d\mu(x)}{\int_{\mathbb{Z}_p} e^{wyt} d\mu(y)} = \sum_{i=0}^{w-1} e^{it} = \sum_{k=0}^{\infty} S_k(w-1) \frac{t^k}{k!} \quad (t \in E). \tag{7}$$

In what follows, we will always assume that the Volkenborn integrals of the various exponential functions on $\mathbb{Z}_p$ are defined for $t \in E$ (cf. (2)), and therefore it will not be mentioned.

[1–3,5,6] are some of the previous works on identities of symmetry in two variables involving Bernoulli polynomials and power sums. For the brief history, one is referred to those papers.

In this paper, we will produce 8 basic identities of symmetry in three variables w_1, w_2, w_3 related to Bernoulli polynomials and power sums (cf. (16),(17),(20),(23),(27),(29),(31),(32)). These and most of their corollaries seem to be new, since there have been results only about identities of symmetry in two variables in the literature. These abundance of symmetries shed new light even on the existing identities. For instance, it has been known that (8) and (9) are equal (cf. [6, Corollary 1], [3, Corollary 2]) and (10) and (11) are so (cf. [3, (13)], [6, Corollary 4]). In fact, (8)–(11) are all equal, as they can be derived from one and the same p-adic integral. Perhaps, this was neglected to mention in [3]. Also, we have a bunch of new identities in (12)–(15). All of these were obtained as corollaries (cf. Corollaries 9, 12, 15) to some of the basic identities by specializing the variable w_3 as 1. Those would not be unearthed if more symmetries had not been available.

$$\sum_{k=0}^{n} \binom{n}{k} B_k(w_1 y_1) S_{n-k}(w_2 - 1) w_1^{n-k} w_2^{k-1}, \tag{8}$$

$$= \sum_{k=0}^{n} \binom{n}{k} B_k(w_2 y_1) S_{n-k}(w_1 - 1) w_2^{n-k} w_1^{k-1}, \tag{9}$$

$$= w_1^{n-1} \sum_{i=0}^{w_1 - 1} B_n(w_2 y_1 + \frac{w_2}{w_1} i), \tag{10}$$

$$= w_2^{n-1} \sum_{i=0}^{w_2 - 1} B_n(w_1 y_1 + \frac{w_1}{w_2} i), \tag{11}$$

$$= \sum_{k+\ell+m=n} \binom{n}{k, \ell, m} B_k(y_1) S_\ell(w_1 - 1) S_m(w_2 - 1) w_1^{k+m-1} w_2^{k+\ell-1}, \tag{12}$$

$$= w_1^{n-1} \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_1 - 1} B_k(y_1 + \frac{i}{w_1}) S_{n-k}(w_2 - 1) w_2^{k-1}, \tag{13}$$

$$= w_2^{n-1} \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_2 - 1} B_k(y_1 + \frac{i}{w_2}) S_{n-k}(w_1 - 1) w_1^{k-1}, \tag{14}$$

$$= (w_1 w_2)^{n-1} \sum_{i=0}^{w_1 - 1} \sum_{j=0}^{w_2 - 1} B_n\left(y_1 + \frac{i}{w_1} + \frac{j}{w_2}\right). \tag{15}$$

The derivations of identities is based on the p-adic integral expression of the generating function for the Bernoulli polynomials in (4) and the quotient of integrals in (7) that can be expressed as the exponential generating function for the power sums. We indebted this idea to the paper [3].

2. Main theorems

Theorem 1. *Let w_1, w_2, w_3 be any positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.*