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Numerically intersecting algebraic varieties via witness sets

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ABSTRACT

The fundamental construct of numerical algebraic geometry is the representation of an irreducible algebraic set, A, by a witness set, which consists of a polynomial system, F, for which A is an irreducible component of $\mathcal{V}(F)$, a generic linear space $\mathcal{L}$ of complementary dimension to A, and a numerical approximation to the set of witness points, $\mathcal{L}\cap A$. Given F, methods exist for computing a numerical irreducible decomposition, which consists of a collection of witness sets, one for each irreducible component of $\mathcal{V}(F)$. This paper concerns the more refined question of finding a numerical irreducible decomposition of the intersection $A\cap B$ of two irreducible algebraic sets, A and B, given a witness set for each. An existing algorithm, the *diagonal homotopy*, computes witness point supersets for $A\cap B$, but this does not complete the numerical irreducible decomposition. In this paper, we use the theory of isosingular sets to complete the process of computing the numerical irreducible decomposition of the intersection.

1. Introduction

This paper concerns the computation of the intersection, $A\cap B$, of two irreducible algebraic sets, A and B. Since the intersection of two general algebraic sets can be broken down into a collection of pair-wise intersections of the irreducible components of those sets, with some membership testing to eliminate multiple appearances of components or strict containment of one in another, the pair-wise intersection of irreducibles is seen to be the crux of a general intersection capability. The method of this paper fits into the framework of numerical algebraic geometry, a term coined in [30]. The area relies on polynomial homotopy algorithms, often called polynomial continuation, to reliably and efficiently compute numerical approximations to the solutions of systems of polynomial equations. Overviews of the field may be found in [31,32]. The basics of polynomial continuation for finding isolated solutions are covered in [15,16,18,31].

In numerical algebraic geometry, an irreducible algebraic set, $A\subset\mathbb{C}^N$, is represented by a *witness set*, W, of the form

$$W=\{F,\mathcal{L},\mathcal{L}\cap A\},\tag{1}$$

where $F:\mathbb{C}^N\to\mathbb{C}^M$ is a polynomial system such that A is an irreducible component of

$$\mathcal{V}(F):=\{x\in\mathbb{C}^N|F(x)=0\},\tag{2}$$

$\mathcal{L}$ is a generic linear space of complementary dimension to A (i.e., $\dim\mathcal{L}=N-\dim A$), and hence, $\mathcal{L}\cap A\subset\mathbb{C}^N$ is a set of $\deg A$ isolated points. Of course, in numerical work, the points $\mathcal{L}\cap A$ are not known exactly but rather are represented by numerical approximations. We call $W=\{F,\mathcal{L},\mathcal{L}\cap A\}$ a *witness set*; it contains a *witness system*, F, and a *witness point set*, $\mathcal{L}\cap A$.

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One of the fundamental operations of numerical algebraic geometry is the computation of a *numerical irreducible decomposition*, which finds a witness set for each irreducible component of $\mathcal{V}(F)$. The first algorithm for computing a numerical irreducible decomposition appeared in [26], with subsequent improvements in [2,27–29]. Let the pure i-dimensional component of $\mathcal{V}(F)$ be denoted Z_i (possibly empty for some i), which decomposes into a finite number, n_i, of distinct irreducible components Z_{ij}; that is,

$$\mathcal{V}(F) = \bigcup_{i=0}^{\dim\mathcal{V}(F)} Z_i, \quad Z_i = \bigcup_{j=1}^{n_i} Z_{ij}. \tag{3}$$

All of the existing algorithms for the irreducible decomposition first compute *witness supersets*, $\widehat{W}_i = \{F, \mathcal{L}_i, \widehat{S}_i\}$, where $\mathcal{L}_i$ is a generic linear space of codimension i and each *witness point superset* $\widehat{S}_i$ is a finite set of points such that $(\mathcal{L}_i \cap Z_i) \subset \widehat{S}_i \subset \mathcal{V}(F)$. Let us denote the local dimension of algebraic set A at z, as $\dim_z A$. To winnow a witness point set from a witness point superset, a filtering algorithm culls out the junk points $J_i = \{x \in \widehat{S}_i | \dim_x \mathcal{V}(F) > i\}$. For points with a low multiplicity bound (obtained as the number of homotopy paths leading to it), this filtering is most efficiently done using a local dimension test [2]; otherwise, homotopy membership tests [27] determine if x is in any component of higher dimension, these having been determined already in sequential order starting at the top dimension. Then the remaining points, $S_i = \widehat{S}_i \setminus J_i$, are partitioned using monodromy [29] and trace tests [28] to form the witness point sets for the irreducible components, $S_{ij} = \mathcal{L}_i \cap Z_{ij}$, thereby forming the witness sets $W_{ij} = \{F, \mathcal{L}_i, S_{ij}\}$. It is possible to do this at each dimension independently, but it is generally more efficient to descend sequentially through all possible values of i using a cascade approach [8,9,22].

Once a witness set for an irreducible algebraic set A is known, one can subsequently apply other operations of numerical algebraic geometry to it. For example, it is possible to test if a given point is in A and, if A is positive dimensional, to compute a scattered sampling of points numerically in A. One can also answer many questions about A, such as its degree and multiplicity structure. Also, given witness sets for two irreducible algebraic sets, one can test if either is contained in the other.

One operation that is incomplete in numerical algebraic geometry is intersection. That is, given witness sets, $\{F_A, \mathcal{L}_A, \mathcal{L}_A \cap A\}$ and $\{F_B, \mathcal{L}_B, \mathcal{L}_B \cap B\}$, for irreducible algebraic sets A and B, we wish to compute the intersection $A \cap B$ as represented by a numerical irreducible decomposition. This would consist of a collection of witness sets, one for each irreducible component of $A \cap B \subset \mathbb{C}^N$. Similar to the case described above for computing a numerical irreducible decomposition of $\mathcal{V}(F)$, let Z_i be the pure i-dimensional component of $A \cap B$, which has an irreducible decomposition $Z_i = \cup_{j=1}^{n_i} Z_{ij}$. For each dimension i where i-dimensional components Z_{ij} of $A \cap B$ could exist, i.e., for $\max\{0, \dim A + \dim B - N\} \leqslant i \leqslant \min\{\dim A, \dim B\}$, existing *diagonal homotopy* methods compute a finite witness point *superset*, $\widehat{S}_i$, such that $(\mathcal{L}_i \cap Z_i) \subset \widehat{S}_i \subset (A \cap B)$, where $\dim \mathcal{L}_i = N - i$. Diagonal methods work by duplicating the variables from, say, $x \in \mathbb{C}^N$, to $(x, y) \in \mathbb{C}^{2N}$, and finding subsets of $A \times B = \{(x, y) | x \in A, y \in B\}$ such that $x = y$. The first such algorithm [24] uses a cascade approach that doubles the number of variables, while a subsequent reformulation [25] in intrinsic coordinates reduces the number of variables.

The computation of the witness point supersets $\widehat{S}_i$ is a crucial first step, but to complete the numerical irreducible decomposition of $A \cap B$, one must eliminate any junk points in the superset to obtain a witness point set for Z_i and then break these set into witness point sets for the irreducible components Z_{ij}. As we shall see, the local dimension test used for eliminating junk in the standard numerical irreducible decomposition problem no longer applies to the pairwise intersection problem. Furthermore, for each irreducible component Z_{ij}, the witness set requires a witness system F_{ij} such that Z_{ij} is one of its irreducible components. In some cases, $F_{ij} = \{F_A, F_B\}$ suffices, but as we shall illustrate below, this is not true in general.

The purpose of this paper is to complete the diagonal approach for computing a numerical irreducible decomposition of a pair-wise intersection. To do so, we provide an alternative approach to filtering out junk points and show how to generate the witness systems F_{ij} to complete the irreducible witness sets. The necessary tools are provided by the recently developed theory of *isosingular sets* [10]. The existing diagonal homotopy suffices as is when all the irreducible components of $A \cap B$ are also irreducible components of $\mathcal{V}(F_A, F_B)$. On the contrary, suppose C is some irreducible component of $A \cap B$ which is not an irreducible component of $\mathcal{V}(F_A, F_B)$. Then, C may or may not be an isosingular set with respect to $\{F_A(x), F_B(x)\}$, but as we shall prove, it is always isomorphic to the intersection of $\mathcal{V}(x - y)$ with an isosingular set of the related system $\{F_A(x), F_B(y)\}$. Using this fact, one can compute a witness set for C and, in this way, complete the numerical irreducible decomposition of $A \cap B$.

While the completion of the diagonal intersection via isosingular theory is the main purpose of this paper, a secondary contribution is to present a simplified formulation of the diagonal homotopy. This formulation is advantageous in the construction of a working computer algorithm.

2. Insufficiency of the diagonal homotopy

Although the existing diagonal homotopies compute witness point supersets for an intersection, this is not enough to produce true witness sets and the irreducible decomposition. As mentioned above, there may exist an irreducible component $C \subset A \cap B$ that is not an irreducible component of the combined system $\{F_A, F_B\}$. A few simple examples follow to illustrate this fact.

Example 2.0.1. Consider the polynomial

$$f(x, y, z) = (x - 1)(x^2 + y^2 + z^2 - 4)((x - 2)^2 + y^2 + z^2 - 4).$$