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Bounds on eigenvalues of real and complex interval matrices

Milan Hladík *

Charles University, Faculty of Mathematics and Physics, Department of Applied Mathematics, Malostranské nám. 25, 11800 Prague, Czech Republic
University of Economics, Faculty of Informatics and Statistics, nám. W. Churchilla 4, 13067 Prague, Czech Republic

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ABSTRACT

We present a cheap and tight formula for bounding real and imaginary parts of eigenvalues of real or complex interval matrices. It outperforms the classical formulae not only for the complex case but also for the real case. In particular, it generalizes and improves the results by Rohn (1998) [5] and Hertz (2009) [19]. The main idea behind is to reduce the problem to enclosing eigenvalues of symmetric interval matrices, for which diverse methods can be utilized.

The result helps in analysing stability of uncertain dynamical systems since the formula gives sufficient conditions for testing Schur and Hurwitz stability of interval matrices. It may also serve as a starting point for some iteration methods.

1. Introduction

Stability of dynamical systems attracts attention of control community for several decades. Discrete dynamical systems lead to a Schur stability, and continuous systems lead to a Hurwitz stability of the system matrices. A matrix A is Schur stable if its spectral radius is less than 1, and A is Hurwitz if real parts of all eigenvalues are negative.

There are intrinsic uncertainties when solving practical problems. Uncertainties are modeled in diverse ways, but interval analysis naturally handles the best and worst cases of continuous domains of parameters. The aim of this paper is to derive cheap but sharp bounds on eigenvalues of complex interval matrices. This will give an efficient tool in stability checking, among others, since we obtain a strong sufficient condition for stability. Thus, we avoid exhaustive and expensive enumerative or branch & bound methods in many cases.

An interval matrix is defined as a family of matrices

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{n\times n};\ \underline{A} \leqslant A \leqslant \overline{A}\},$$

where $\underline{A},\ \overline{A} \in \mathbb{R}^{n\times n},\ \underline{A} \leqslant \overline{A}$, are given matrices, and the inequality is considered element-wise. The midpoint and the radius of $\boldsymbol{A}$ are denoted respectively by

$$A_c := \frac{1}{2}(\underline{A} + \overline{A}), \quad A_\Delta := \frac{1}{2}(\overline{A} - \underline{A}).$$

The set of all $n \times n$ interval matrices is denoted by $\mathbb{IR}^{n\times n}$. A complex interval matrix as a family $\boldsymbol{A} + i\boldsymbol{B}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are interval matrices of order n. The eigenvalue set $\Lambda(\boldsymbol{A} + i\boldsymbol{B})$ corresponding to $\boldsymbol{A} + i\boldsymbol{B}$ is defined as the set of all eigenvalues over all $A + iB \in \boldsymbol{A} + i\boldsymbol{B}$, that is,

* Address: Charles University, Faculty of Mathematics and Physics, Department of Applied Mathematics, Malostranské nám. 25, 11800 Prague, Czech Republic.

E-mail addresses: milan.hladik@matfyz.cz, hladik@kam.mff.cuni.cz

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$$\Lambda(\boldsymbol{A}+i\boldsymbol{B}) = \{\lambda + i\mu \mid \exists A \in \boldsymbol{A}\ \exists B \in \boldsymbol{B}\ \exists x + iy \neq 0 : (A+iB)(x+iy) = (\lambda + i\mu)(x+iy)\}.$$

Next, we use $\rho(A)$ for the spectral radius of A. The real and imaginary part of a complex number z is denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

Deif [1] presents a description of the eigenvalue sets and their exact bounds, however, it is valid only under some assumptions on sign pattern invariance of eigenvectors, which are not so easy to verify. In praxis, it mostly suffices to calculate a fast computable enclosures (supersets) of eigenvalue sets. Kolev and Petrakieva [2] develop an enclosure for real parts eigenvalues by solving nonlinear system of equations; an exact bound can be achieved under some monotonicity assumptions. Kolev [3] extends it to the class of interval parametric matrices. Mayer [4] proposes an enclosure method for eigenvalues of real and complex interval matrices based on Taylor expansion. A cheap formula for an enclosure is in Rohn [5]. An estimation on eigenvalues based on perturbation theory appears in Ahn et al. [6].

Even though complex matrices are less common in practice, there are still some applications and techniques using them, and thus motivation our research. Stability of systems with complex matrices is studied e.g. in [7–9]. Ahn et al. [10] reduced the problem of checking robust stability of a fractional-order linear time invariant uncertain interval system to finding maximal eigenvalues of a Hermitian complex interval matrix.

For Hurwitz stability checking, Franze et al. [11] present a sufficient condition by using a Gershgorin-type theorem. Xiao and Unbehauen [12] show that Schur/Hurwitz stability checking can be reduced to checking only exposed faced of an interval matrix. Further, Rohn [13] proved that Hurwitz stability can be reduced to inspecting 2^{2n-1} special vertex matrices provided that each matrix in $\boldsymbol{A}$ has real eigenvalues only. Stability analysis based on Lyapunov equation was studied in [10,14,15].

A special subclass of interval matrices are symmetric interval matrices. For an interval matrix $\boldsymbol{A}$, the corresponding symmetric interval matrix $\boldsymbol{A}^S$ is defined as a family of all symmetric matrices in $\boldsymbol{A}$, that is,

$$\boldsymbol{A}^S = \{A \in \boldsymbol{A} \mid A = A^T\}.$$

A real symmetric matrix $A \in \mathbb{R}^{n\times n}$ has n real eigenvalues; we can assume that they are sorted in a non-increasing order as follows

$$\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A).$$

Extending the notation for symmetric interval matrices, we introduce

$$\lambda_i(\boldsymbol{A}^S) = [\underline{\lambda}_i(\boldsymbol{A}^S), \overline{\lambda}_i(\boldsymbol{A}^S)] := \left\{\lambda_i(A) \mid A \in \boldsymbol{A}^S\right\}, \quad i = 1,\ldots,n$$

the eigenvalue sets of a symmetric interval matrix $\boldsymbol{A}^S$. The eigenvalue sets represent n compact intervals, which are either disjoint or may overlap [16].

Even though the main focus of this paper is on bounding eigenvalues of general (complex) interval matrices, we review some methods for symmetric interval matrices since our main result reduces computation of the general case to the symmetric one. Various bounding methods are discussed in Hladík et al. [16]. For checking Schur/Hurwitz stability a symmetric interval matrix, [17] proposed a branch & bound algorithm. Other related results are found in [18–21].

The following simple but efficient bounds are due to Rohn [21].

Theorem 1 (Rohn, 2005). *For each $i \in \{1,\ldots,n\}$ one has*

$$\boldsymbol{\lambda}_i(\boldsymbol{A}^S) \subseteq [\lambda_i(A_c) - \rho(A_\Delta), \lambda_i(A_c) + \rho(A_\Delta)].$$

As shown by Hertz [18,19], the extremal eigenvalue limits $\overline{\lambda}_1(\boldsymbol{A}^S)$ and $\underline{\lambda}_n(\boldsymbol{A}^S)$ can be computed exactly by inspecting 2^{n-1} special vertex matrices.

Theorem 2 (Hertz, 1992). *Define $Z := \{1\} \times \{\pm 1\}^{n-1} = \{(1, \pm 1, \ldots, \pm 1)\}$ and for a $z \in Z$ define $A_z,\ A\prime_z \in \boldsymbol{A}^S$ in this way:*

$$(a_z)_{ij} = \begin{cases} \overline{a}_{ij} & \text{if } z_i = z_j, \\ \underline{a}_{ij} & \text{if } z_i \neq z_j, \end{cases} \quad (a\prime_z)_{ij} = \begin{cases} \underline{a}_{ij} & \text{if } z_i = z_j, \\ \overline{a}_{ij} & \text{if } z_i \neq z_j. \end{cases}$$

Then

$$\overline{\lambda}_1(\boldsymbol{A}^S) = \max_{z\in Z} \lambda_1(A_z), \quad \underline{\lambda}_n(\boldsymbol{A}^S) = \min_{z\in Z} \lambda_n(A\prime_z).$$

The main focus of this paper is on bounding complex eigenvalues of interval matrices. One of the basic bounds is the following formula by Rohn [5].

Theorem 3 (Rohn, 1998). *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. Then for each eigenvalue $\lambda + i\mu \in \Lambda(\boldsymbol{A})$ we have*

$$\lambda \leqslant \lambda_1\left(\frac{1}{2}(A_c + A_c^T)\right) + \rho\left(\frac{1}{2}(A_\Delta + A_\Delta^T)\right),$$