Applied Mathematics and Computation 218 (2012) 11466–11471

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# The Ramanujan master theorem and its implications for special functions

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ARTICLE INFO

Keywords:
Ramanujan master theorem
Special functions
Bessel functions
Integration methods
Umbral methods

ABSTRACT

We study a number of possible extensions of the Ramanujan master theorem, which is formulated here by using methods of Umbral nature. We discuss the implications of the procedure for the theory of special functions, like the derivation of formulae concerning the integrals of products of families of Bessel functions and the successive derivatives of Bessel type functions. We stress also that the procedure we propose allows a unified treatment of many problems appearing in applications, which can formally be reduced to the evaluation of exponential- or Gaussian-like integrals.



## 1. Introduction

The Ramanujan master theorem (RMT) [1–3] states that if the function $f(x)$ is defined through the series expansion

$$f(x)=\sum_{r=0}^{\infty}\frac{(-x)^r}{r!}\varphi(r) \tag{1}$$

with $\varphi(0)\neq 0$ then the following identity holds

$$\int_0^{\infty}x^{\nu-1}f(x)dx=\Gamma(\nu)\varphi(-\nu). \tag{2}$$

A proof, albeit not rigorous, of the theorem can be achieved by the use of the Umbral methods [4], namely by setting [3,5]

$$f(x)=\sum_{r=0}^{\infty}\frac{(-x)^r}{r!}\hat{c}^r\varphi(0), \tag{3}$$

with the operator $\hat{c}$ defined by:

$$\hat{c}^r\varphi(0)=\varphi(r). \tag{4}$$

In this way the function $f(x)$ can be formally written as an pseudo-exponential function, $f(x)=e^{-\hat{c}x}\varphi(0)$, and thus the integral in Eq. (2) can be given in the form

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http://dx.doi.org/10.1016/j.amc.2012.05.036

$$\int_0^\infty x^{\nu-1}e^{-\hat{c}x}dx\varphi(0) = \Gamma(\nu)\hat{c}^{-\nu}\varphi(0), \tag{5}$$

and using Eq. (4) we end up with the result quoted in Eq. (2). For further comments see the concluding Section and for a rigorous proof Refs. [1,2].

An immediate consequence of the theorem is the evaluation of the following integral

$$I_{\nu,\alpha} = \int_0^\infty x^{\nu-1}C_\alpha(x)dx, \tag{6}$$

where

$$C_\alpha(x) = \sum_{r=0}^\infty \frac{(-x)^r}{r!\Gamma(r+\alpha+1)} \tag{7}$$

is the Tricomi–Bessel function of order $\alpha$ [6] which satisfies the conditions of the RMT, with $\varphi(0) = 1/\Gamma(\alpha+1)$. According to Eq. (2), we get

$$I_{\nu,\alpha} = \frac{\Gamma(\nu)}{\Gamma(\alpha-\nu+1)}. \tag{8}$$

The procedure we have just quoted, which traces back to Crofton [3] and to other operationalists, is by no means a proof of the theorem but just a guiding tool, which will be proved to be very useful for the forthcoming speculations.

As the function $C_\alpha(x)$ is linked to the ordinary Bessel functions by the identity

$$C_\alpha(x) = (2x)^{-\alpha/2}J_\alpha(2\sqrt{x}), \tag{9}$$

the use of the RMT may be of noticeable interest for the evaluation of various integrals appearing in applications.

As noted in Ref. [5] the Umbral formalism may be useful to make some progress towards an extension of the RMT theorem. We observe indeed that if

$$g_m(x) = e^{-\hat{c}x^m}\varphi(0) = \sum_{r=0}^\infty \frac{(-x^m)^r}{r!}\varphi(r), \tag{10}$$

by applying the same procedure as in Eqs. (5), we have

$$\int_0^\infty dx x^{\nu-k}g_m(x) = \frac{1}{m}\Gamma\left(\frac{\nu+1-k}{m}\right)\varphi\left(-\frac{\nu+1-k}{m}\right). \tag{11}$$

Just as an application of this result, we note that integrals of the type

$$G_0(b) = \int_{-\infty}^{+\infty} e^{bx}g_2(x)dx, \tag{12}$$

can be written as *pseudo*-Gaussian integral of the form

$$G_0(b) = \int_{-\infty}^\infty e^{bx}e^{-\hat{c}x^2}dx\varphi(0) = \sqrt{\frac{\pi}{\hat{c}}}e^{b^2/(4\hat{c})}\varphi(0) = \sqrt{\pi}\sum_{r=0}^\infty \frac{b^{2r}}{4^r r!}\hat{c}^{-r-1/2}\varphi(0) = \sqrt{\pi}\sum_{r=0}^\infty \frac{b^{2r}}{4^r r!}\varphi(-r-1/2). \tag{13}$$

In the case of $\varphi(r) = 1/r!$, Eq. (13) becomes

$$G_0(b) = \int_{-\infty}^\infty e^{bx}J_0(2x)dx = \sqrt{\pi}\sum_{r=0}^\infty \frac{b^{2r}}{4^r r!}\frac{1}{\Gamma(-r+1/2)} = \frac{2}{\sqrt{4+b^2}}. \tag{14}$$

Further examples will be discussed later in this paper, which is devoted to understand a variety of consequence emerging from RMT and from its Umbral revisitation.

In Section 2 we will explore its relevance in the field of special functions, while in Section 3 we will discuss some concluding remarks.

## 2. The RMT and special functions

Let us consider the integral (2) in which the function $f(x)$ is given by

$$f(x) = e^{-ax-bx^2}. \tag{15}$$

This function satisfies the RMT conditions, since it can be expanded as follows

$$f(x) = \sum_{n=0}^\infty \frac{(-x)^n}{n!}H_n(a,-b), \tag{16}$$