Applied Mathematics and Computation 218 (2012) 10094–10105

Contents lists available at SciVerse ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

The $K(m,n)$ equation with generalized evolution term studied by symmetry reductions and qualitative analysis

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ARTICLE INFO

Keywords:
The $K(m,n)$ equation
Symmetry reductions
Hamiltonian system
Conserved quantity
Solitary, kink and periodically traveling wave solutions

ABSTRACT

In this paper we obtain symmetry reductions of the $K(m,n)$ equation with generalized evolution term. The reduction to ordinary differential equations comes from an optimal system of subalgebras. Some of these equations admit symmetries which lead to further reductions, and one of them comes out suitable for qualitative analysis. Its dynamical behavior is fully described and conservative quantities are stated.

1. Introduction

The $K(m,n)$ equation with generalized evolution term, was introduced by Biswas in [1] and it is given by

$$(u^l)_t + au^m u_x + b(u^n)_{xxx} = 0, \tag{1.1}$$

where $a, b \in \mathbb{R}^*$ and $l, m, n \in \mathbb{Z}^+$. The first term is the generalized evolution term, the second and the third terms represent the convection one and the dispersion one, respectively.

In [2], Bruzón and Gandarias presented a procedure to look for exact solutions of nonlinear ordinary differential equations (ODE's), which leads to solutions (not obtained in [1]) in terms of Jacobi elliptic functions for specific values of the parameters l, m, n, a and b of Eq. (1.1). This equation is a generalized form of the $K(m,n)$ equation, usually introduced as

$$u_t + a(u^m)_x + b(u^n)_{xxx} = 0 \tag{1.2}$$

and, in turn, of the Korteweg–de Vries (KdV) equation, where $l = m = n = 1$. On the other hand, Eq. (1.1) is equivalent to

$$v_t + \frac{a}{l} v^{\frac{m+1-l}{l}} v_x + b(v^{\frac{n}{l}})_{xxx} = 0,$$

after using the transformation $u = v^{\frac{1}{l}}$, so it is sufficient to consider the case $l = 1$ if just $\frac{m+1-l}{l}; \frac{1}{l} \in \mathbb{Z}^+$.

Different variants or particular cases of the $K(m,n)$ equation are found in the literature [2–12]. Recently Chen and Li [3] have studied the simple peak solitary wave solutions of the osmosis $K(2,2)$ equation under the inhomogeneous boundary conditions and they have obtained all smooth, peaked and cusped solitary wave solutions of it. The modified KdV (mKdV) equations (Eq. (1.1) with $l = 1,\ m = 2$ and $n = 1$) and their solutions have also been studied intensively. Liu and Li ([4] and its references) considered an extended form of the mKdV equation of the form

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2012.03.084

$$u_t + a_1 u_{xxx} + a_2 u_x + a_3 u u_x + a_4 u^2 u_x = 0,$$

the all exact solutions based on the Lie group method were given, and the bifurcations and traveling wave solutions were obtained. Rosenau and Hyman [5] studied the role of nonlinear dispersion in the formation of patterns in liquid drops of the nonlinear dispersive equations.

$$u_t + u^m u_x + (u^n)_{xxx} = 0$$

for $m > 0,\ 1 < n \leqslant 3$. They also introduced a class of solitary wave solutions with compact support, i.e. solutions with absence of infinite wings or absence of infinite tails, called *compactons*. In addition to compactons, Rosenau [6] proved that the nonlinear dispersive equations $K(m,n)$

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0,$$

which exhibit a number of remarkable dispersive effects, can support both: kinks and solitons with an infinite slope(s), periodic waves and dark solitons with cusp(s), all being manifestations of nonlinear dispersion in action. For $n < 0$ the enhanced dispersion at the tail may generate algebraically decaying patterns. Other solitary-wave solutions of $K(m,n)$ equations were also found by Rosenau in [7,8].

It is known that many integrable equations arise naturally from motions of plane or space curves. In [9,10] the authors investigated the possibility that the $K(m+1,m)$ and $K(m+2,m)$ models can be obtained from plane curves in certain geometries, which provides the geometric interpretations to $K(m,n)$ equations.

Existing techniques for solving nonlinear partial differential equations (PDE's) include: Inverse scattering transform, Wadati trace method, pseudo–spectral method, tanh–sech method, sine–cosine method, Riccati equation expansion method, exponential function method, etc. ([1] and references within it). In spite of the key role of these particular techniques used for solving the equations, one of their limitations is that they do not lay down the conserved quantities. This drawback is, for example, partially overcome in [1], where a 1-soliton solution of Eq. (1.1) is obtained by using the solitary wave ansatz, and a conserved quantity is calculated. Among the techniques, the methods of point transformations are a powerful tool. By means of the Theory of Symmetry Reductions [13,14] a single group reduction may transform a PDE with two independent variables into ODE's. Local symmetries admitted by a PDE are useful for finding invariant solutions. These solutions are obtained by using group invariants to reduce the number of independent variables. The basic idea of the technique is that, a reduction transformation exists when a differential equation is invariant under a Lie group of transformations. The machinery of the Lie group theory provides a systematic method to search for these special group invariant solutions. Although symmetry constraints are powerful in determining integrability of PDE's, not all of them yield exact solutions of the equations, as pointed out in [15].

It is an interesting and important problem how to generally explore integrability of nonlinear PDE's by integrable ODE's. There is a pretty general scheme to reduce PDE's into integrable ODE's. The separation of the time and space variables without using any structure associated with evolution equations is analyzed in [16], and an extension by means of the Frobenius integrable decompositions (FID) is introduced for partial differential equations in [17]. The resulting theory provides techniques which are applied in particular to the celebrated KdV and MKdV equations. The resulting integrable decompositions have exhibited many interesting solution relations with integrable ODE's, including those relations of traveling wave solutions with scalar differential equations and one-dimensional Hamiltonian systems. It also generalizes the Theory of Symmetry constraints in soliton theory, since it does not require any structure associated with the equations under investigation, such as Lax pairs for soliton equations and the symmetry property in symmetry constraints.

The dynamical systems theory [18–20] provides fundamental tools for dealing with ODE's, by qualitative analysis and conservative quantities. Previous works have used them to deal with ODE's coming from PDE's problems. In [21], solutions that present behaviors like sources, asymptotic plane waves, and blow up process at finite time have been characterized. In [22], singular perturbation theory has been applied for analyzing the solutions. In several works, Tang et al. have studied the traveling wave solutions of a given PDE according to different parametric conditions. In [11,12,23,24], bifurcations of phase portraits are discussed in detail, and although the conservative aspects of the system are not dealt with, a first integral (conserved quantity) is deduced. In particular, this study is applied to a generalized KdV equation in [12] and to $K(n,-n,2n)$ equations in [11].

In this work we consider the $K(m,n)$ equation with generalized evolution term (1.1). The paper is organized as follows: first, a complete calculus of the different reductions admitted by this equation is developed (Sections 2 and 3). Second, among the reduced equations, the most general case comes out suitable for qualitative analysis. Indeed, the reduced equation yields to a conservative system, and this allows us to make a complete characterization of its possible dynamical behaviors (Section 4).

2. Classical symmetries

To apply the classical method to Eq. (1.1) with $a, b \neq 0$ we consider the one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \end{aligned}$$