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Modified HSS iteration methods for a class of non-Hermitian positive-definite linear systems ☆

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A R T I C L E   I N F O

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A B S T R A C T

We consider the numerical solution of a class of non-Hermitian positive-definite linear systems by the modified Hermitian and skew-Hermitian splitting (MHSS) iteration method. We show that the MHSS iteration method converges unconditionally even when the real and the imaginary parts of the coefficient matrix are nonsymmetric and positive semidefinite and, at least, one of them is positive definite. At each step the MHSS iteration method requires to solve two linear sub-systems with real nonsymmetric positive definite coefficient matrices. We propose to use inner iteration methods to compute approximate solutions of these linear sub-systems. We illustrate the performance of the MHSS method and its inexact variant by two numerical examples.

1. Introduction

Many problems in scientific computing require to solve the linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a large sparse non-Hermitian positive definite matrix and $x, b \in \mathbb{C}^n$. Here we use A^H to denote the conjugate transpose of the matrix A, and we call a non-Hermitian matrix $A \in \mathbb{C}^{n\times n}$ positive definite if its Hermitian part $\frac{1}{2}(A + A^H)$ is positive definite; see [1]. For solving this class of linear systems, a sequence of splitting iteration methods has been developed, e.g., *Hermitian and skew-Hermitian splitting* (HSS) iteration [8], *preconditioned Hermitian and skew-Hermitian splitting* (PHSS) iteration [6], and *block triangular and skew-Hermitian splitting* (BTSS) iteration [7]; see [4,5,10–17] for other developments and [2] for an overview.

To solve the linear system (1.1) iteratively, Bai et al. [8] used the *Hermitian and skew-Hermitian splitting* (HSS) of the coefficient matrix A, i.e.,

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^H) \quad \text{and} \quad S = \frac{1}{2}(A - A^H)$$

and established the following HSS iteration method:

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Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \dots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where α is a given positive constant.

In [8] Bai et al. also proved that for any positive constant α the HSS iteration method converges unconditionally to the unique solution of the linear system (1.1). In [9] Bai, Golub and Ng presented the *inexact Hermitian and skew-Hermitian splitting* (IHSS) scheme to avoid the exact inverses of the n-by-n matrices $\alpha I + H$ and $\alpha I + S$. They also proved that the asymptotic convergence rate of the IHSS iteration approach to that of the HSS iteration when the tolerances of the inner iterations tend to zero as the number of outer iteration steps increases.

Denote by $\imath$ the imaginary unit. When $A \in \mathbb{C}^{n\times n}$ has the form $A = W + \imath T$, where $W, T \in \mathbb{R}^{n\times n}$ are real symmetric matrices, with W positive definite and T positive semidefinite, we can see that the Hermitian part H and the skew-Hermitian part S of A become

$$H = W \quad \text{and} \quad S = \imath T$$

and the HSS iteration scheme becomes

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - \imath T)x^{(k)} + b, \\ (\alpha I + \imath T)x^{(k+1)} = (\alpha I - W)x^{(k+\frac{1}{2})} + b. \end{cases}$$

To avoid the complex arithmetic, Bai et al. [3] skillfully modified the above HSS iteration scheme, and presented the following *modified Hermitian and skew-Hermitian splitting* (MHSS) iteration method:

Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \dots$ until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + W)x^{(k+\frac{1}{2})} = (\alpha I - \imath T)x^{(k)} + b, \\ (\alpha I + T)x^{(k+1)} = (\alpha I + \imath W)x^{(k+\frac{1}{2})} - \imath b, \end{cases} \tag{1.2}$$

where α is a given positive constant.

In [3] Bai et al. proved that the MHSS iteration method converges unconditionally to the unique solution of the linear system for any positive constant α. Also, they showed that when

$$\alpha = \sqrt{\gamma_{min}\gamma_{max}},$$

an upper bound of the spectral radius of the MHSS iteration matrix can be minimized, and the corresponding minimum upper bound is given by $\frac{\sqrt{\kappa(W)+1}}{\sqrt{\kappa(W)}+1}$, where γ_{min} and γ_{max} are the minimum and the maximum eigenvalues of the matrix W, respectively, and $\kappa(W) = \gamma_{max}/\gamma_{min}$ is the spectral condition number of W.

In this paper, we consider the MHSS iteration method for solving the linear system (1.1) with $A = W + \imath T$, where $W, T \in \mathbb{R}^{n\times n}$ are real nonsymmetric matrices. We prove that the MHSS iteration method also converges to the unique solution of the linear system mentioned above for any positive parameter α.

The paper is organized as follows. In Section 2, we discuss the convergence property of the MHSS iteration method for non-Hermitian positive definite linear systems. In Section 3, we present the IMHSS iteration method. Finally, in Section 4 we use numerical examples to illustrate the effectiveness of our methods.

2. Convergence analysis of MHSS iteration

In this section, we study the convergence property of the MHSS iteration method under certain conditions on the matrices W and T.

The MHSS iteration scheme (1.2) can be equivalently rewritten as

$$x^{(k+1)} = M(\alpha)x^{(k)} + G(\alpha)b, \quad k = 0, 1, 2, \dots,$$

where

$$M(\alpha) = (\alpha I + T)^{-1}(\alpha I + \imath W)(\alpha I + W)^{-1}(\alpha I - \imath T)$$

and

$$G(\alpha) = (1 - \imath)\alpha(\alpha I + T)^{-1}(\alpha I + W)^{-1}.$$

Thus, $M(\alpha)$ is the iteration matrix of the MHSS iteration method. Actually, we can split A into

$$A = B(\alpha) - C(\alpha),$$